Parimutuel Betting under Asymmetric Information[1]

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Abstract

This paper examines finite parimutuel betting games with asymmetric information, with particular attention to differences between sequential and simultaneous settings, and between fully rational and myopic ("price taking") behavior. In the simultaneous parimutuel market, all (symmetric and asymmetric) Bayesian-Nash equilibria are generically characterized depending on the number of bettors and the quality of their private information. There always exists a separating equilibrium, where all bettors follow their private signal. This equilibrium becomes unique as the number of bettors increases, and it corresponds to the strategy profile used by myopic bettors. In the sequential framework, the perfectly revealing equilibrium disappears as the number of betting periods increases, whether or not bettors fully anticipate their impact on future odds. In both cases (rational and myopic betting), due to the interaction between information externalities generated by observational learning and payoff externalities generated by betting odds, bettors arbitrate between following their private signal, following the choices of previous bettors, and betting against the trend. Extreme effects based on herd behavior occur in identifiable states of the world, leading to significant short run mispricing.

KEYWORDS: Parimutuel betting; Asymmetric information; Information aggregation; Herd behavior; Contrarian behavior.

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1 Introduction

Goldman Sachs and Deutsche Bank recently introduced options on economic data releases such as employment, retail sales, industrial production, inflation, and economic growth which provide a way of hedging core risk, i.e., they allow firms and individuals to share the risk of uncertain economic outcomes.[1] The first auction took place on October 1, 2002 for options on the September U.S. nonfarm payrolls report. Building on this initial success,[2] the two pioneering investment banks have since then added options auctions based on, among others, the monthly report of the Institute of Supply Management manufacturing, the US Retail Sales report (excluding cars), and the Eurozone Harmonized Index of Consumer Prices (excluding tobacco). These new economic derivatives, which have been advocated for a long time by economists such as Robert Shiller (see Shiller, 1995), are priced and allocated *parimutuelly* meaning that their prices are based solely on the relative demand for their implied outcomes. According to Goldman Sachs and Deutsche Bank this pricing mechanism is transparent and fair, and Lange and Economides (2004) show that it offers many advantages over the dominant dealer and exchange continuous time mechanisms. Among other advantages, the parimutuel mechanism removes the requirement of a discrete order match between two counterparties and creates liquidity in otherwise illiquid markets such as contingent claims on corporate earnings.

The origin of the parimutuel system lies in horse-race betting. In 1865, a Frenchman named Pierre Oller, in reaction to losing too much of his own money to the bookmakers, developed a wagering system which dispenses with odds makers who use their judgment to decide how much a given wager should pay. He called his system “parier mutuel”, meaning “mutuel stake” or “betting among ourselves.” When the system was adopted in England, it became known as “Paris mutuals” and then “pari-mutuels.” Nowadays parimutuel wagering is the accepted betting procedure at major horse-racing tracks throughout the world; it is exclusively used by racetracks in North America, France, Hong Kong, and Japan and it coexists with a bookmaking market in Australia and Great Britain.[3] In these parimutuel gambling systems, if the horse chosen by an individual wins, then his investment yields returns (calculated through final betting odds) that are decreasing with the proportion of bettors who have bet on the same horse. Besides, the proportion of the money in the win pool that is bet on any given horse is interpreted as the subjective probability that this horse will win the race and it corresponds to the *implicit price* of one unit of money bet. Several theoretical and empirical studies in economics have investigated the features of parimutuel betting markets and have pointed out their particular relevance for the analysis of decision making under risk and for the analysis of market efficiency.[4]

In this paper we propose a game-theoretical analysis of parimutuel betting markets with asymmetric information. Such markets are characterized by well defined interactive decision

[1] These options also provide publicly available information about the likelihood of these outcomes.

[2] See the Financial Times, 24-25 May 2003.

[3] Parimutuel gambling is also offered at greyhound racing, jai-alai frontons, lotto games, etc.

[4] A detailed account of this recently developed literature may be found in Sauer (1998) and Vaughan Williams (1999).

situations which possess several features of commonly analyzed games. First, they are anonymous games in the sense that a player's payoff only depends on the distribution of strategies over the set of players rather than on other players' identities. Second, they are characterized by negative payoff externalities as the payoff of each player decreases with the number of other players who bet on the same option. Third, they are "almost" n-person zero-sum games since, except in the case where all bettors choose the incorrect option, the return to each individual is a fractional proportion of the entire amount wagered on the market (in particular, Pareto efficiency problems are excluded). Finally, they are characterized by a well-defined termination point at which the value of each bet becomes certain. In consequence, the analysis of parimutuel betting markets can provide a clear view of pricing issues and information aggregation which are more complicated elsewhere.

Almost all the few existing theoretical studies on racetrack parimutuel betting have focused on games with symmetric information, either by considering bettors with homogeneous beliefs (Chadha and Quandt, 1996; Feeney and King, 2001), or by introducing uninformed bettors modelled as noise bettors (Hurley and McDonough, 1995; Terrell and Farmer, 1996; Koessler, Ziegelmeyer, and Broihanne, 2003), or by endowing bettors with inconsistent beliefs (Watanabe, Nonoyama, and Mori, 1994; Watanabe, 1997). Contrary to those previous contributions, we consider strategic parimutuel betting markets where differences in beliefs are due only to differences in information, and are not arbitrary unexplained differences in opinions. Those differences in information may be due, e.g., to the dispersion of knowledge concerning the intrinsic ability of each horse, the condition of the track, the skill of each jockey, horses' performances in previous races, etc. In such a setting, we especially wish to analyze the relative effectiveness of parimutuel betting markets at aggregating information from diverse sources. In a theoretical analysis related to ours, Ottaviani and Sørensen (2003) also consider strategic parimutuel betting markets with asymmetrically informed bettors and provide explanations for two robust empirical regularities, namely the favorite-longshot bias and the timing of informative bets. Wherever possible, we compare our results to theirs throughout the following sections.

In compliance with standard game theory, we first study parimutuel betting under asymmetric information by assuming fully rational bettors. Those are standard rational players that incorporate in their optimization problem their beliefs about predecessors' and successors' strategies and beliefs. In that way, they take into account their full impact on the equilibrium betting odds. In a second step, we consider bettors with myopic foresight who do not try to predict the future bets. More precisely, we study parimutuel betting under asymmetric information with *boundedly rational* or *myopic* players who only consider *current betting odds* and only evaluate their *own* impact on them but neglect the signaling effect of their bets on future bets. However, myopic bettors still deduce information from the possible observation of odds movements.

In order to isolate the effect of payoff externalities we begin to consider as a benchmark framework the **simultaneous move version of the game**. Generically (for almost all bettors' qualities of information), we characterize the set of all (symmetric and asymmetric) pure strategy Bayesian-Nash equilibria according to the number of bettors and the quality of their private information. In particular, we show that a separating equilibrium where all bettors use their private information always exists. Asymmetric equilibria where some players bet non-informatively on the same horse whatever their private signal also exist. Though such a multiplicity of equilibria represents a serious problem and raises selection questions, we show that such a selection is needless for a large number of bettors since asymmetric equilibria vanish when the number of bettors increases. Consequently, in large simultaneous wagering markets odds against each horse always reflect all private information. Interestingly enough, myopic bettors' optimal behavior agrees with the strategy used by rational bettors in the separating equilibrium. Indeed, since betting odds are undetermined at the moment of their choice, horses have the same return, which implies that each myopic bettor simply follows his private signal in the simultaneous betting market.

Next, we investigate **sequential betting** in an attempt to capture the strategic behavior of static (simultaneous) betting games, the social learning of sequential trade models, and signaling behavior. In such a setting, bets, in themselves, could reveal the underlying information and so affect the behavior of latter bettors. Therefore, the effect of information on odds takes an added dimension as the possibility of multiple rounds of bets is considered. By relying on rational bettors, we show that a perfectly revealing equilibrium can fail to exist, and will typically disappear when the number of bettors increases. Intuitively, this failure might be related to the fact that bettors exhibit *herd behavior*, as in standard models with information cascades (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992). Such behavior arises if later decision makers choose to blindly imitate their predecessors' choice without heed to their own private information. However, as in most multi-agent sequential trade models with asymmetric information, the price mechanism ensures that complete imitation does not hold in the long-run as bettors arbitrate between following their private signal, following the choices of previous bettors (herd behavior), and betting *against* the trend (*contrarian behavior*).[5] Our analysis of parimutuel betting markets with rational asymmetrically informed bettors leads us to conclude that there is a clear-cut distinction between the equilibrium of large simultaneous betting markets and large sequential betting markets with odds against each horse reflecting all private information in the former equilibrium but not in the latter one.

A similar conclusion is drawn when the analysis is conducted under the assumption of myopic bettors who are not able to anticipate the future fluctuation of betting odds. In the sequential betting market, myopic bettors are almost "price takers" since they only anticipate their own impact on the current odds, but ignore others' current and future impact on them. Myopic betting still remains more complex than in the simultaneous framework as myopic bettors might deduce previous bettors information from the observation of the past dynamic of betting odds. With the help of simulation results we show that, as with fully rational bettors, odds adjustments do not prevent herd and contrarian behavior because betting odds

[5] For a survey on sequential trade models with asymmetric information see, e.g., Brunnermeier (2001).

do not integrate all information available in the history of bets. In particular, herd behavior leads to extreme short run effects in some states of the world.[6]

The paper proceeds as follows: Section 2 lays out the framework and basic model of parimutuel betting used throughout the paper. In Section 3, we cover the results concerning the simultaneous move betting game. Section 4 investigates the sequential move version of the game. We provide some concluding remarks and we discuss the relation to previous studies in Section 5. The Appendix contains some proofs.

2 A Model of Parimutuel Betting

In this section we present our general modelling setup of racetrack parimutuel betting. As most theoretical studies in this literature, we consider a horse race where there are only two horses. Additionally, we do not allow bettors to withdraw from betting and we assume that each bettor is only able to bet one unit of money. Said differently, bettors must spend on the race their entire endowment of one unit of money.[7] Finally, we consider a finite set of asymmetrically informed bettors. All of them are assumed to integrate the direct consequences of their actions on the odds of the gamble, and they are endowed with beliefs about the state of Nature obtained from their signals and from the information contained in the choices of their predecessors. Parimutuel betting under asymmetric information will be studied by considering both fully rational bettors and myopic bettors. Fully rational bettors are standard rational players that incorporate in their optimization problem their beliefs about predecessors and successors' strategies and beliefs. In that way, they take into account their full impact on the equilibrium betting odds. On the contrary, myopic bettors consider current betting odds and only evaluate their own impact on them. However, they still deduce information from the possible observation of odds movements.

We investigate parimutuel betting under asymmetric information when bettors are either endowed with rational or myopic expectations. Accordingly, we can evaluate whether the degree of far-sightedness of the bettors has an impact on the relative effectiveness of parimutuel betting markets at aggregating information from diverse sources. Studying parimutuel betting under asymmetric information by considering myopic behavior enables us also to directly compare our results to those obtained in the information cascades literature and especially

[6]It is worth mentioning that the model and results presented in this paper can be applied to a variety of problems of investment decisions in which the payoff of each agent is decreasing with the number of agents who make the same decision and where individuals are asymmetrically informed about the return of each alternative. In particular, our study can be applied to models of congestion where the utility of a product like a transportation device, a firm's location, or a visit of a public event, decreases with the quantity of users. For example, as suggested by Feeney and King (2001), bettors in parimutuel markets may be assimilated to investors who have to choose between several towns in which to locate similar retail stores. Introducing asymmetric information allows one to analyze a situation in which investors are privately informed about, e.g., the state of demand in each town.

[7]If fully rational bettors were able to drop from betting, then a no-trade result would apply and the only equilibrium would be the trivial equilibrium where all bettors refrain from betting. A complementary analysis to the one presented in this paper would be to investigate rational parimutuel betting with withdrawing possibilities by adding players who bet for exogenous reasons.

to Avery and Zemsky's (1998) results. Such a thorough comparison is performed in the last section of the paper.

Independently of its temporal structure, the parimutuel betting market can be modelled as follows. The two *horses* are called A and B. There is a finite set $N = \{1, \ldots, n\}$ of *risk neutral bettors*. Each bettor $i \in N$ independently chooses to bet one unit of money on a horse $s_i \in S_i \equiv \{A, B\}$. For any vector of bets $s = (s_1, \ldots, s_n) \in S = \prod_{i \in N} S_i$ and any horse $H \in \{A, B\}$, let $H(s) = \{i \in N : s_i = H\}$ be the set of bettors who bet on horse H, let $h(s) = |H(s)|$ be the number of bettors who bet on horse H, and let $\overline{H}$ be the horse other than H. The *odds against horse* H, which is given by the total number of bets on horse $\overline{H}$ divided by the total number of bets on horse H, is denoted by

$$O_H(s) = \frac{n - h(s)}{h(s)}.$$

We assume that bettors have a common prior belief on the set of states of Nature $\Theta = \{\theta_A, \theta_B\}$, where θ_A stands for "horse A wins" and θ_B stands for "horse B wins". For simplicity, we consider a flat prior belief: $\Pr(\theta_A) = \Pr(\theta_B) = 1/2$. The results would be essentially the same with unequal prior winning chances.[8]

If bettor i bets on the winning horse, then his payoff is normalized to the return of this horse, which is equal to its odds plus 1. Otherwise, bettor i receives 0 payoff. Hence, the payoff of each bettor depends on his own choice, on alternatives chosen by other bettors, and on the actual state of Nature. For example, odds of 4 to 1 laid against a horse implies a payoff to a successful bet of four units of money, plus the stake returned (which is equal to one unit of money here); on the contrary, an unsuccessful bet loses the stake. Formally, each bettor $i \in N$ has a (vNM) *utility function* $u_i : S \times \Theta \to \mathbb{R}$ such that for all $H \in \{A, B\}$,

$$u_i(s, \theta) = \begin{cases} O_H(s) + 1 = \frac{n}{h(s)} & \text{if } s_i = H \text{ and } \theta = \theta_H \\ 0 & \text{if } s_i = H \text{ and } \theta \neq \theta_H. \end{cases} \tag{1}$$

Before taking his decision, and in addition to the possible history of choices he has observed, each bettor $i \in N$ gets a *private signal* $q_i \in Q_i \equiv \{q^A, q^B\}$ that is correlated to the true state of Nature. Conditionally on the state of Nature, bettors' signals are i.i.d. and satisfy

$$\begin{aligned} \Pr(q_i = q^A \mid \theta_A, q_j) &= \Pr(q_i = q^A \mid \theta_A) = \pi > 1/2 \\ \Pr(q_i = q^A \mid \theta_B, q_j) &= \Pr(q_i = q^A \mid \theta_B) = 1 - \pi, \end{aligned}$$

for all i, $j \in N$, $i \neq j$. Hence, once bettor i has received a signal $q_i \in Q_i$, his beliefs about the states of Nature are given by $\Pr(\theta_A \mid q^A) = \Pr(\theta_B \mid q^B) = \pi$ and $\Pr(\theta_A \mid q^B) = \Pr(\theta_B \mid q^A) =$

[8] For example, if bets are made simultaneously and if we consider different prior beliefs for horses A and B, then, in asymmetric equilibria, the number of non-informative bets on each horse are not necessarily the same. Actually, it can be shown that in such equilibria the number of unconditional bets on horse A are increasing with the prior probability of horse A, and inversely for horse B. We do not consider those generalizations since they introduce integer problems which complicate the exposition without substantially modifying the results.

$1-\pi$. The parameter π characterizes bettors' *quality* or *precision of information*. The larger is π, the larger is the precision of bettors' signals.

3 Simultaneous Betting

To avoid the complexities associated with social learning and signaling of sequential games, we first analyze the simultaneous parimutuel betting game. In such a setting, a bettor's strategy is a rule of action which maps each realization of his signal to one of two actions: to bet on horse A, or to bet on horse B. More precisely, a (pure) *strategy* of bettor i is a mapping $\sigma_i : Q_i \to S_i$. Bettor i's belief that the winning horse is horse H and that other bettors receive the vector of signals q_{-i} is given by

$$\Pr(\theta_H, q_{-i} \mid q_i) = \Pr(q_{-i} \mid \theta_H) \Pr(\theta_H \mid q_i) = \left(\prod_{j \neq i} \Pr(q_j \mid \theta_H) \right) \Pr(\theta_H \mid q_i). \tag{2}$$

Thus, using Equations (1) and (2), bettor i's *expected utility* given others' strategies σ_{-i} when he bets on horse H and receives the signal q_i can be written

$$U_i(H, \sigma_{-i} \mid q_i) = \Pr(\theta_H \mid q_i) \sum_{q_{-i} \in Q_{-i}} \left(\prod_{j \neq i} \Pr(q_j \mid \theta_H) \right) \frac{n}{h(H, \sigma_{-i}(q_{-i}))}.$$

As usual, in a Bayesian-Nash equilibrium the action prescribed by each bettor's strategy maximizes his expected payoff conditional on his signal given others' strategies. More precisely, a *Bayesian-Nash equilibrium*, or simply equilibrium, is a profile of strategies σ such that for all $i \in N$, $q_i \in Q_i$, and $s_i \in S_i$ we have $U_i(\sigma_i(q_i), \sigma_{-i} \mid q_i) \geq U_i(s_i, \sigma_{-i} \mid q_i)$.[9] The following lemma establishes that there is no equilibrium where a bettor systematically bets against his private signal.

Lemma 1 *For any number of bettors, n, and any quality of information, π, the strategy which involves bettor i in betting against his private signal, i.e., $\sigma_i(q^H) = \overline{H}$ for all $H \in \{A, B\}$, is strictly dominated.*

Proof. The strategy $\sigma_i(q^H) = \overline{H}$ for all H is optimal for bettor i if $U_i(\overline{H}, \sigma_{-i} \mid q^H) \geq U_i(H, \sigma_{-i} \mid q^H)$ for all H, which is impossible with $\pi > 1/2$. □

On the contrary, a natural and intuitive strategy for a bettor is simply to follow his private signal, i.e., to bet on horse A if he receives the signal q^A and to bet on horse B if he receives the signal q^B. In the following proposition we show that there is always an equilibrium where all bettors use this "informative" strategy, and this equilibrium is the unique symmetric one.

[9] Since we deal in this paper only with pure strategy equilibria, no confusion shall arise when we write "equilibrium" instead of "pure strategy equilibrium". To simplify the exposition, we ignore indifference cases throughout the paper when they are due to non-generic values of π.

We call this equilibrium the *separating equilibrium.* In such an equilibrium, betting odds perfectly reflect all private information.

Proposition 1 *For any number of bettors, n, and any quality of information, π, there exists an equilibrium where every bettor follows his private signal, i.e., $\sigma_i(q^H) = H$ for all $H \in \{A, B\}$ and $i \in N$. This equilibrium is the unique symmetric equilibrium.*

Proof. Let $i \in N$ and assume that $\sigma_j(q^H) = H$ for all $H \in \{A, B\}$ and $j \neq i$. Then, for any $H \in \{A, B\}$, inequality $U_i(H, \sigma_{-i} \mid q^H) \geq U_i(\overline{H}, \sigma_{-i} \mid q^H)$ is satisfied if and only if $\Pr(\theta_H \mid q^H) \geq \Pr(\theta_{\overline{H}} \mid q^H)$, i.e., $\pi \geq 1 - \pi$, which is satisfied since $\pi > 1/2$. To prove that the separating equilibrium is the unique symmetric equilibrium, remark that there are three other symmetric strategy profiles: (i) every bettor bets against his own private signal, (ii) every bettor bets on horse A, and (iii) every bettor bets on horse B. We prove that none of those strategy profiles form a Bayesian-Nash equilibrium.

(i) Directly from Lemma 1.

(ii) Let $\sigma_i(q^H) = A$ for all $H \in \{A, B\}$ and $i \in N$. Then, $U_i(B, \sigma_{-i} \mid q^B) = \Pr(\theta_B \mid q^B)\frac{n}{1} = n\pi$ and $U_i(A, \sigma_{-i} \mid q^B) = \Pr(\theta_A \mid q^B)\frac{n}{n} = (1 - \pi) < n\pi$. Hence, σ is not an equilibrium.

(iii) Similar to (ii). □

Now, we investigate the possibility of asymmetric equilibria. For any non-negative integers j and k, the binomial coefficient is denoted by $C_k^j \equiv \frac{k!}{j!(k-j)!}$.

Proposition 2 *There exists an equilibrium where exactly $k \in \{0, 1, \ldots, n-2\}$ bettors always follow their private signal if and only if $n - k$ is an even number, $\frac{n-k}{2}$ bettors always bet on horse A, $\frac{n-k}{2}$ bettors always bet on horse B, and the following inequality is satisfied:*

$$\frac{\pi}{1-\pi} \leq \sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2} + j} \Big/ \sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2} + 1 + j}. \tag{3}$$

If such an equilibrium exists, then there are exactly $C_n^k \times C_{n-k}^{\frac{n-k}{2}}$ equilibria with exactly k informative bets.

Proof. See the Appendix. □

An equilibrium with exactly k informative bets is called a *k-informative equilibrium.* For example, if $k = 0$ and n is an even number, then from the previous proposition we get an equilibrium where half bettors always bet on horse A and half bettors always bet on horse B if and only $\pi \leq \frac{n+2}{2n+2}$. If $k = 1$ and n is an odd number, then we get an equilibrium if $(n-1)/2$ bettors always bet on horse A, $(n-1)/2$ bettors always bet on horse B, and $\pi \leq \frac{1}{1+\sqrt{\frac{n-1}{n+3}}}$.

We are not able to explicitly solve Inequality (3) for π with arbitrarily values of k. However, we provide in the following proposition an interesting property of the solution. This proposition states that the polynomial in π induced by Inequality (3) has one and only one real root on the interval $]1/2, 1[$. More precisely, as in the previous exemples with $k = 0$ and

$k = 1$, partially informative equilibria as described in Proposition 2 exist only for sufficiently small qualities of information.

Proposition 3 *For all $n \geq 2$ and $k < n$ such that $n - k$ is an even number, there exists $\pi(k, n) \in]1/2, 1[$ (the maximum quality of information for a k-informative equilibrium to exist) such that Inequality (3) is satisfied if and only if $\pi \leq \pi(k, n)$.*

Proof. See the Appendix. □

It can be numerically verified that $\pi(k, n)$ is increasing in k and decreasing in n.[10] That is, conditions for the existence of a k-informative equilibrium are weaker as the number of informative bets, k, increases, but become stronger as the total number of bettors increase. The next proposition shows that the separating equilibrium is in fact unique in large enough parimutuel betting markets.

Proposition 4 *For all $\pi > 1/2$, there exists $\overline{n}$ such that for all $n \geq \overline{n}$ the unique Bayesian-Nash equilibrium of the n-player simultaneous game is the separating equilibrium.*

Proof. See the Appendix. □

3.1 Simultaneous Betting with Myopic Bettors

In the simultaneous parimutuel betting game, myopic bettors' optimal behavior is a trivial single-decision problem: since betting odds are undetermined at the moment of their choice, horses have the same return, so each myopic bettor simply follows his private signal. Such a behavior is similar to the "informative" strategy adopted by rational bettors in the separating equilibrium (see Proposition 1). Accordingly, in large betting markets, myopic bettors' behavior is identical to fully rational bettors' behavior and leads to betting odds perfectly reflecting all private information.

3.2 The Favorite-Longshot Bias

Even though betting odds perfectly reflect all private information in large parimutuel markets, the winning chances of the favorite are always underestimated by the equilibrium/myopic distribution of bets. More precisely, when the number of bettors increases, the equilibrium/myopic market's estimates of the winning chances of the favorite tend towards the quality of bettors' private information (viz., the probability that any bettor receives a correct signal, which is lower than one but greater than one-half) whereas the objective winning probability of the favorite that could be deduced from the observation of the final equilibrium/myopic distribution of bets tends towards one. This phenomenon, called the *favorite-longshot bias*, is a well known empirical regularity of parimutuel betting markets, and implies that a bettor

[10] This has been checked numerically up to $n = 100$ bettors for all possible values of k.

who is able to bet after all other bettors can make abnormal earnings by simply betting on the horse with the largest frequency of bets (the favorite).

In a richer setting with continuous signal spaces, Ottaviani and Sørensen (2003) also establish that the winning chances of the favorite are underestimated by the distribution of bets resulting from the unique symmetric equilibrium (no characterization of the asymmetric equilibria is provided).

In this section we have assumed that bettors' decisions were made simultaneously. The results obtained here provide a benchmark against which to compare equilibrium/myopic outcomes for the case where we allow sequential betting. This latter framework is more appealing since in modern parimutuel betting markets high-speed electronic calculators, known as totalizators or tote boards, record and display up-to-the-minute betting patterns. In a similar vein, detailed pricing, volatility, and probability information is provided electronically in real time during the economic derivatives auctions launched by Deutsche Bank and Goldman Sachs.

In this sequential move version of the market, we will show that a perfectly revealing equilibrium does not always exist because bettors use observed odds to deduce information of previous bettors. As will become clear later, those information externalities generated by previous players' bets induce subsequent bettors to arbitrate between following their private signal and following previous bettors' decisions. As parimutuel betting involve negative payoff externalities, players might also go against the observed trend whatever their private information. Thus, the next section investigates how such information is revealed by the price process, and, at the same time, to which extent does this publicly revealed information affect bettors' behaviors.

4 Sequential Betting

While in most equilibria of the simultaneous betting game odds against horses reflect all players' private information, bettors cannot use this publicly revealed information. Hence, all dimensions of information aggregation can only be examined in a sequential framework. In this respect, we explore in this section the predictions of our model under a more general assumption regarding observability of actions by letting bets chosen by any player to be observable by the rest of the market.

4.1 The Dynamic Parimutuel Game

At each stage in the sequence, bettors can observe the bets cast by preceding bettors, but not the signals that those earlier bettors received. A *history* of k decisions is denoted by $s^k \in S^k = \prod_{j=1}^{k}\{A, B\}$. When bettor $k+1$ takes a decision, he observes a history s^k, i.e., he observes the decisions taken by bettors $1, \ldots, k$. Let $H(s^k) = \{i \in \{1, \ldots, k\} : s_i = H\}$ be the set of bettors up to period k who chose horse H, and let $h(s^k) = |H(s^k)|$ be the associated number of bets on horse H.

A (pure) *strategy* of bettor i is a function $\sigma_i : S^{i-1} \times Q_i \to S_i$. Hence, $\sigma_i(s^{i-1}; q_i)$ is the choice of bettor i with signal q_i when he has observed the history s^{i-1}. A profile of strategies is denoted by $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma = \prod_{i\in N} \Sigma_i$. Denote respectively by $q_{i^+} = (q_j)_{j>i}$ and $\sigma_{i^+} = (\sigma_j)_{j>i}$ the signals and strategies of the successors of bettor i, and let $Q_{i^+} = \prod_{j>i} Q_j$ and $\Sigma_{i^+} = \prod_{j>i} \Sigma_j$. Let $s(\sigma_{i^+} \mid s^i; q_{i^+})$ be the final history reached according to the profile of strategies $\sigma_{i^+} \in \Sigma_{i^+}$ given the history $s^i \in S^i$ and the profile of signals $q_{i^+} \in Q_{i^+}$. Let $s(\sigma \mid q) = s(\sigma \mid \emptyset; q)$ be the final history reached according to the profile of strategies σ given the profile of signals q, and let $s(\sigma_{n^+} \mid s^n; q_{n^+}) = s^n$.

Bettor i's *belief* about the state of Nature θ when he has received the signal q_i and has observed the history s^{i-1} is denoted by $\mu_i(\theta \mid s^{i-1}; q_i)$. The conditional independence of the signals gives

$$\Pr(\theta_H, q_{i^+} \mid s^{i-1}, q_i) = \Pr(q_{i^+} \mid \theta_H)\, \mu_i(\theta_H \mid s^{i-1}; q_i). \tag{4}$$

Bettor i's expected utility is obtained from Equation (4) and from bettors' utility function (see Equation (1) on page 5):

$$U_i(H, \sigma_{i^+} \mid s^{i-1}; q_i) = \mu_i(\theta_H \mid s^{i-1}; q_i)\ \mathrm{E}\left(\frac{n}{h(s(\sigma_{i^+} \mid s^{i-1}, H; q_{i^+}))} \;\middle|\; \theta_H \right). \tag{5}$$

As usual, a pair (σ, μ) is a *perfect Bayesian equilibrium* if μ is obtained from σ by Bayes' rule whenever possible (weak consistency), and if σ satisfies sequential rationality. Formally, a pair (σ, μ) is *sequentially rational* if for all $i \in N$, $q_i \in Q_i$, and $s^{i-1} \in S^{i-1}$ we have $U_i(\sigma_i(s^{i-1}; q_i), \sigma_{i^+} \mid s^{i-1}; q_i) \geq U_i(s_i, \sigma_{i^+} \mid s^{i-1}; q_i)$, for all $s_i \in S_i$. For all $j \in N$, let $J_j(s_j \mid s^{j-1}) \equiv \{q_j \in Q_j : \sigma_j(s^{j-1}; q_j) = s_j\}$ be the set of signals such that bettor j plays s_j after he has observed the history $s^{j-1} \in S^{j-1}$. If $J_j(s_j \mid s^{j-1}) \neq \emptyset$ for all $j < i$, then bettor i can apply Bayes' rule: $\mu_i(\theta \mid s^{i-1}; q_i) = \Pr(\theta \mid q_i, q_j \in J_j(s_j \mid s^{j-1})\ \forall\ j < i)$. Hence, a pair (σ, μ) is *weakly consistent* if for all $i \in N$, $q_i \in Q_i$, and $s^{i-1} \in S^{i-1}$ we have

$$\mu_i(\theta \mid s^{i-1}, q_i) = \frac{\Pr(q_i \mid \theta) \prod_{j<i} \sum_{q_j \in J_j(s_j \mid s^{j-1})} \Pr(q_j \mid \theta)}{\sum_{\theta_H \in \{\theta_A, \theta_B\}} \Pr(q_i \mid \theta_H) \prod_{j<i} \sum_{q_j \in J_j(s_j \mid s^{j-1})} \Pr(q_j \mid \theta_H)}, \tag{6}$$

whenever $J_j(s_j \mid s^{j-1}) \neq \emptyset$ for all $j < i$.

The next lemma extends Lemma 1 to the dynamic game : no player has an incentive to systematically bet against his private signal, the reason being essentially that they can bet only in one period, so they are not able to manipulate the market.

Lemma 2 *For any number of bettors, n, and any quality of information, π, the strategy which involves bettor i in betting against his private signal for some history s^{i-1}, i.e., $\sigma_i(s^{i-1}; q^H) = \overline{H}$ for all $H \in \{A, B\}$, is strictly dominated.*

Proof. Assuming that $U_i(B \mid s^{i-1}, q^A) \geq U_i(A \mid s^{i-1}, q^A)$ and $U_i(A \mid s^{i-1}, q^B) \geq U_i(B \mid s^{i-1}, q^B)$ with $\pi > 1/2$ yields to a contradiction, as in the simultaneous game. □

The next lemma will be useful in several steps of our analysis:

Lemma 3 *Bettor n bets on horse H if and only if $\mu_n(\theta_H \mid s^{n-1}; q_n) \geq \frac{h(s^{n-1})+1}{n+1}$.*

Proof. Directly from the expected utility of Equation (5) on the page before. □

In the next subsection we characterize the set of perfectly revealing equilibria. Contrary to the simultaneous setting we show that perfectly revealing equilibria do not always exist in a sequential betting market, and they disappear as the number of bettors increases.

4.2 Perfectly Revealing Equilibria

From Lemma 2 we know that an equilibrium strategy σ_i for player i is perfectly revealing if and only if player i always follows his own private signal, whatever his signal and whatever the history he has observed (betting against his private signal whatever his signal cannot be part of an equilibrium). More generally, we say that a profile of strategies up to player $i-1$, $(\sigma_i)_{i<j}$, is perfectly revealing if $\sigma_i(s^{i-1}; q^H) = H$ for all $i < j$, $s^{i-1} \in S^{i-1}$, and $H \in \{A, B\}$.

Perfectly revealing equilibria are relatively easy to analyze because an individual's deviation does not influence the actions of later bettors. The next lemma characterizes bettors' updated beliefs when previous bettors used perfectly revealing strategies.

Lemma 4 *If the profile of strategies $(\sigma_j)_{j<i}$ is perfectly revealing then*

$$\mu_i(\theta_H \mid s^{i-1}; q^H) = \frac{\pi^{2h+2-i}}{\pi^{2h+2-i} + (1-\pi)^{2h+2-i}},$$
$$\text{and} \quad \mu_i(\theta_H \mid s^{i-1}; q^{\overline{H}}) = \frac{\pi^{2h-i}}{\pi^{2h-i} + (1-\pi)^{2h-i}},$$

for all $H \in \{A, B\}$ and $s^{i-1} \in S^{i-1}$, where $h = h(s^{i-1})$.

Proof. From Bayes rule (Equation (6) on the preceding page) with $J_j(H \mid s^{j-1}) = \{q^H\}$ (because every player follows his signal by assumption), after some simplifications. □

The next proposition shows that there is a range of parameter values for which perfectly revealing betting fails to be an equilibrium. More precisely, it is shown that if the quality of bettors' private information is not fairly balanced then a perfectly revealing equilibrium does not exist. This is a simple but nice result since it neatly categorizes when the payoff externality and the information externality cancel each other. The intuition is the following. Consider the last bettor in the sequence (bettor n). If the quality of information is too low and the private signal of bettor n is in accordance with previous decisions, then the majority of past bets on the favorite is not sufficiently convincing that the favorite will really win, and thus it is optimal for bettor n to bet against the trend and his own private signal. Such a behavior is called a *contrarian behavior* as the agent ignores his private signal and bets against the current favorite. On the contrary, if the quality is too high, then previous bets provide strong evidence that the favorite will win, and thus bettor n will not follow his signal if it favors the longshot. This behavior is called *herd behavior* as the agent follows the trend independently of his private signal. In this later case, the fact that previous bettors have bet on the same horse reveals relevant information about the winning chance of this horse (the favorite). Such public information may overwhelm the bettor's private assessment and may counterbalance the negative effect of the odds against the favorite (which is low), and thus may cause a bettor otherwise inclined to bet on the longshot to bet on the horse chosen by previous bettors.

Proposition 5 *Let $n > 2$. If (σ, μ) is a perfectly revealing equilibrium then*

$$\frac{n^{1/(n-2)}}{1 + n^{1/(n-2)}} \geq \pi \geq \frac{n^{1/n}}{1 + n^{1/n}}.$$

Proof. Let (σ, μ) be a perfectly revealing equilibrium. This implies that the last bettor always follows his signal: $\sigma_n(s^{n-1}; q^H) = H$ for all $s^{n-1} \in S^{n-1}$. In particular, (i) $\sigma_n(\overline{H}, \ldots, \overline{H}; q^H) = H$ and (ii) $\sigma_n(H, \ldots, H; q^H) = H$.

(i) From Lemmas 3 and 4 we have ($h = 0$): $\mu_n(\theta_H \mid s^{n-1}; q^H) \geq \frac{h(s^{n-1})+1}{n+1} \Leftrightarrow \frac{\pi^{2-n}}{\pi^{2-n}+(1-\pi)^{2-n}} \geq \frac{1}{n+1} \Leftrightarrow \frac{n^{1/(n-2)}}{1+n^{1/(n-2)}} \geq \pi$.

(ii) From Lemmas 3 and 4 we have ($h = n-1$): $\mu_n(\theta_H \mid s^{n-1}; q^H) \geq \frac{h(s^{n-1})+1}{n+1} \Leftrightarrow \frac{\pi^n}{\pi^n+(1-\pi)^n} \geq \frac{n}{n+1} \Leftrightarrow \pi \geq \frac{n^{1/n}}{1+n^{1/n}}$. This completes the proof. □

This necessary condition for a perfectly revealing equilibrium to exist becomes more and more restrictive as the number of bettors, n, increases.[11] Consequently, when the number of players grew larger, there is a clear-cut distinction between the equilibrium sets of the static and dynamic markets.

While a perfectly revealing equilibria usually fail to exist, bettors will doubtlessly follow their own signal when the majority of previous decisions is not too pronounced. That is, the informativeness of bets crucially depends on the observed history. A combination of uninformative bets (herd and contrarian bets) and informative bets may constitute an equilibrium. In the next subsection we characterize such equilibria with two and three bettors.

4.3 Characterization of Equilibria with Two and Three Bettors

The next Proposition 6 shows that in the two periods sequential game the first player always bets accordingly to his signal. On the contrary, the strategy of the bettor who takes his decision in the second position depends on the quality of information. If the quality is low, then he always plays the current longshot (contrarian behavior). Otherwise, if the quality is high, then he always follows his signal.

Proposition 6 *Consider the 2-bettor sequential betting game. There exists a perfectly revealing equilibrium if and only if $\pi \geq \frac{\sqrt{2}}{1+\sqrt{2}}$. Otherwise, there is an equilibrium where the first*

[11] It is not difficult to provide a necessary and *sufficient* condition for a perfectly revealing equilibrium to exist (see Koessler and Ziegelmeyer, 2001, Proposition 7). This condition is slightly stronger than the condition of the previous proposition but it is intractable (it cannot be analytically solved in π for arbitrary values of n).

bettor always follows his signal and the second bets against the first bettor's choice whatever his signal. Those equilibria are sequential equilibria (in the sense of Kreps and Wilson, 1982), and for generic values of π *there is no other sequential equilibrium.*

Hence, herd behavior is not possible with only two bettors. The reason is that two different signals cancel each other out and, since there is a negative payoff externality, the second bettor never follows the first bettor by betting against his own private signal. When the parimutuel betting market involves more than two players we will see that past histories can overwhelm later bettors' signals if π is relatively high which entails the occurrence of herding behavior. The following proposition characterizes the set of all equilibria in which the first two bettors always follow their private signals.

Proposition 7 *Consider the 3-bettor sequential betting game. Assume that the first and the second bettors always follow their private signals.*

- *If* $1 - 2\Big(\frac{2}{3(\sqrt{177}-9)}\Big)^{1/3} + \frac{(\frac{1}{2}(\sqrt{177}-9))^{1/3}}{3^{2/3}} < \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}$, *then* σ *is an equilibrium iff the third bettor uses a contrarian behavior, i.e.,* $\sigma_3(H, H; q_3) = \overline{H}$ *for all* $q_3 \in Q_3$ *and* $\sigma_3(H, \overline{H}; q^H) = \sigma_3(\overline{H}, H; q^H) = H$ *for all* $H \in \{A, B\}$.
- *If* $\frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} < \pi < 3/4$, *then* σ *is an equilibrium iff the third bettor always follows his signal, i.e.,* σ *is perfectly revealing.*
- *If* $\pi > 3/4$, *then* σ *is an equilibrium iff the third bettor herds, i.e.,* $\sigma_3(H, H; q_3) = H$ *for all* $q_3 \in Q_3$ *and* $\sigma_3(H, \overline{H}; q^H) = \sigma_3(\overline{H}, H; q^H) = H$ *for all* $H \in \{A, B\}$.

Contrary to the 2-bettor case, herd behavior is now possible when the quality of information is high. The behavior of the third bettor in the equilibria described by Proposition 7 is relatively intuitive. If the quality of information is low, then the negative effect of the odds against the favorite dominates the belief that this horse will win, and thus the third bettor bets on the longshot whatever his private information. If the quality of information is high, then the third bettor strongly believes that the favorite will win, and thus he bets on the favorite and also disregards his private information. For intermediate qualities of information, neither information externalities nor payoff externalities dominate, and thus the third bettor follows his signal.

There are other equilibria than those described by Proposition 7. Unfortunately, those equilibria may coincide, i.e., different types of equilibria may exist for generic qualities of information. For example, it can be shown that if $\pi \leq \frac{\sqrt{2}}{1+\sqrt{2}}$, then there is an equilibrium where the first bettor follows his signal and the other bettors adopt a contrarian behavior. For higher number of bettors, a plethora of other equilibria coexist. In the next subsection we consider less sophisticated bettors for whom unique dynamic behaviors can be characterized.

4.4 Dynamic of Bets with Myopic Bettors

In this subsection we consider that the parimutuel betting market is composed of *myopic* players who do not anticipate the signaling effect of their bets on future bets. In particular, they believe that odds resulting from their bets are the final odds. Nonetheless, while myopic bettors are not endowed with forward looking behaviors, they still perform backward looking behaviors. That is, myopic bettors maximize a particular perceived payoff and they update their information via Bayes' rule, believing that all previous bettors are myopic bettors.

Considering myopic bettors has at least four advantages for the analysis of parimutuel betting markets. First, equilibrium outcomes are unique, indifference cases notwithstanding. Second, and related, we are able to provide an exhaustive analysis of bettors' behavior in markets with a significant number of bettors. Third, it enables us to see whether equilibrium predictions, both in the static and in the dynamic models, are robust to the behavioral assumptions. Finally, direct comparisons with previous studies on herd behavior in financial markets where no lookahead behavior is required can be made (see Section 5).

While optimal strategies are easier to compute with myopic bettors, we will see that equilibrium predictions are very close to those with standard rational assumptions. For exemple, conditions for the existence of a perfectly revealing equilibrium are similar to those with fully rational bettors. Additionally, herd and contrarian behaviors also arise in the sequential model, so the most important and interesting phenomena obtained before are not lost here.

The new sequential game with myopic bettors is modeled as follows. Given a sequence of bets up to player i, $s^i \in S^i$, the *odds of period* i against horse H is

$$O_H(s^i) = \frac{i - h(s^i)}{h(s^i)}.$$

For all H, the (perceived) *payoff* of myopic bettor i is

$$u_i(s^i, \theta) = \begin{cases} O_H(s^i) + 1 = \frac{i}{h(s^i)} & \text{if } s_i = H \text{ and } \theta = \theta_H \\ 0 & \text{if } s_i = H \text{ and } \theta \neq \theta_H. \end{cases}$$

So defined, myopic bettors incorporate the direct effect of their choices on the current betting odds, but not on later odds movements.

In the sequential model the information structure and beliefs are defined exactly as in the previous section but the (perceived) expected utility takes a quite simpler form because current bettors do not forecast future bettors' behavior:

$$U_i(H \mid s^{i-1}; q_i) = \mu_i(\theta_H \mid s^{i-1}; q_i) \frac{i}{1 + h(s^{i-1})}, \tag{7}$$

where, as before, $\mu_i(\theta \mid s^{i-1}, q_i)$ is given by Equation (6).[12] For all H, we obtain

$$U_i(H \mid s^{i-1}; q_i) > U_i(\overline{H} \mid s^{i-1}; q_i) \text{ iff } \mu_i(\theta_H \mid s^{i-1}; q_i) > \frac{h(s^{i-1}) + 1}{i+1}, \tag{8}$$

which corresponds to Lemma 3 with $i = n$.

Since rewards decrease with the number of individuals having chosen the correct option, it is easy to see that long-run herding is impossible in parimutuel betting markets. Indeed, if long-run herding would occur in some period k, then all bettors after some history $s^k \in S^k$ would bet on some horse H independently of their private signal. In that case, beliefs would remain constant after period k, whereas betting odds against horse H would continuously decline. Therefore, there is unavoidably a period after period k in which a bettor switches by betting on the other horse, a contradiction with the fact that they all follow the crowd. Nevertheless, as informational externalities coexist with negative payoff externalities, the emergence of pronounced herd behavior cannot be prevented.

As in the previous section, we can investigate conditions for the existence of a perfectly revealing equilibrium. Similar conditions are found; in particular, when the number of bettors increase, the perfectly revealing equilibrium disappears. Indeed, the necessary condition given by Proposition 5 also applies here since this condition was found by considering the last bettor, who is in the same situation in both behavioral models.[13]

Without necessarily assuming perfectly revealing strategies, the form of bettor i's posterior probability that horse H wins after inferring, from the history s^{i-1}, $k_H(s^{i-1}) = k_H$ signals favoring horse H (q^H signals) and $k_{\overline{H}}(s^{i-1}) = k_{\overline{H}}$ signals favoring horse $\overline{H}$ ($q^{\overline{H}}$ signals) is

$$\mu_i(\theta_H \mid s^{i-1}; q_i) = \begin{cases} \frac{\pi^{k_H+1}(1-\pi)^{k_{\overline{H}}}}{\pi^{k_H+1}(1-\pi)^{k_{\overline{H}}} + (1-\pi)^{k_H+1}\pi^{k_{\overline{H}}}} & \text{if } q_i = q^H \\ \frac{\pi^{k_H}(1-\pi)^{k_{\overline{H}}+1}}{\pi^{k_H}(1-\pi)^{k_{\overline{H}}+1} + (1-\pi)^{k_H}\pi^{k_{\overline{H}}+1}} & \text{if } q_i = q^{\overline{H}}, \end{cases} \tag{9}$$

where we necessarily have $k_H + k_{\overline{H}} \leq i - 1$. Of course, $\mu_i(\theta_H \mid s^{i-1}; q_i)$ is increasing with the difference $k_H - k_{\overline{H}}$, and $k_H = k_{\overline{H}}$ implies $\mu_i(\theta_H \mid s^{i-1}; q_i) = 1/2$. The procedure we used to compute the optimal strategy of each bettor depending on his signal and the history he has observed is the following.

Bettor 1. Let $k_H(\emptyset) = k_{\overline{H}}(\emptyset) = 0$. For all $q_1 \in Q_1$ and $H \in \{A, B\}$, we obtain $\mu_1(\theta_H \mid q_1)$ from Equation (9) and $\sigma_1(q_1)$ from the best response deduced from Inequality (8). Of course, we get here $\mu_1(\theta_H \mid q^H) = \pi$, $\mu_1(\theta_H \mid q^{\overline{H}}) = 1 - \pi$, and $\sigma_1(q^H) = H$.

[12] Of course, outside equilibrium beliefs are irrelevant for the analysis of equilibrium outcomes with myopic bettors.

[13] Sufficient conditions are slightly stronger than those with fully rational bettors (see Koessler and Ziegelmeyer, 2001, for more details). The intuition is that the strongest conditions are imposed on the last bettor and, at the perfectly revealing equilibrium, the last bettor is exactly in the same situation when all bettors are myopic as when all bettors are fully rational.

Bettor i. For all $s^{i-1} \in S^{i-1}$ and $H \in \{A, B\}$, let

$$k_H(s^{i-1}) = \begin{cases} k_H(s^{i-2}) + 1 & \text{if } \sigma_{i-1}(s^{i-2}; q^H) = s_{i-1} \text{ and } \sigma_{i-1}(s^{i-2}; q^{\overline{H}}) \neq s_{i-1} \\ k_H(s^{i-2}) & \text{otherwise.} \end{cases}$$

We obtain $\mu_i(\theta_H \mid s^{i-1}; q_i)$ from Equation (9) and $\sigma_i(s^{i-1}; q_i)$ from the best response deduced from Inequality (8).

Computing this procedure for given qualities of information $\pi \in]0.5, 1[$ allows us to obtain beliefs and optimal strategies of each bettor in the sequence for all histories and all profiles of signals.[14] We have represented the set of all possible outcomes for the first five bettors in Figures 1, 2, and 3, where in the first column "a" ("b", resp.) in position $i \in \{1, \ldots, 5\}$ corresponds to a signal q^A (q^B, resp.) for bettor i.[15] As with fully rational bettors, we see that there is a preponderance of uninformative bets with low qualities of information due to contrarian behaviors and with high qualities of information due to herd behavior. All bets are always informative only for qualities of information between approximately 0.59 and 0.63. Hence, at least in the short-run, contrarian and herd behaviors arise whether or not bettors are endowed with forward looking behavior. In the next subsection we illustrate how herd behavior can lead to disproportionate mispricing up to at least 15 periods.

[14]The computational program is available from the authors upon request.

[15]When a bettor is indifferent between betting on horse A and betting on horse B we assume as a tie-breaking rule that he bets on horse B.

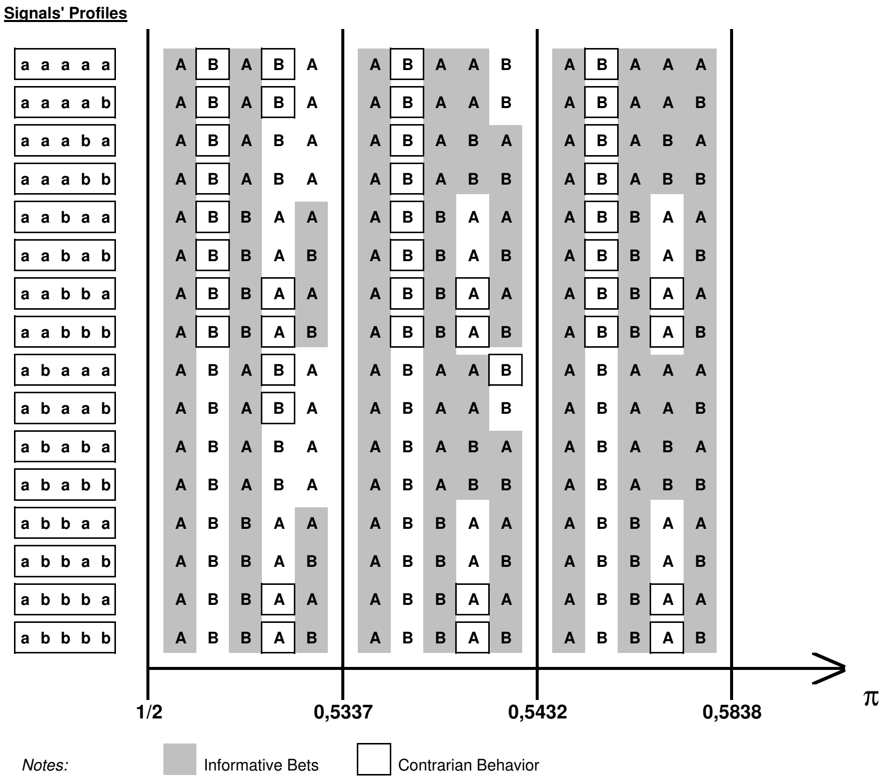

Figure 1: Set of Outcomes for the First Five Myopic Bettors with Low Values of π.

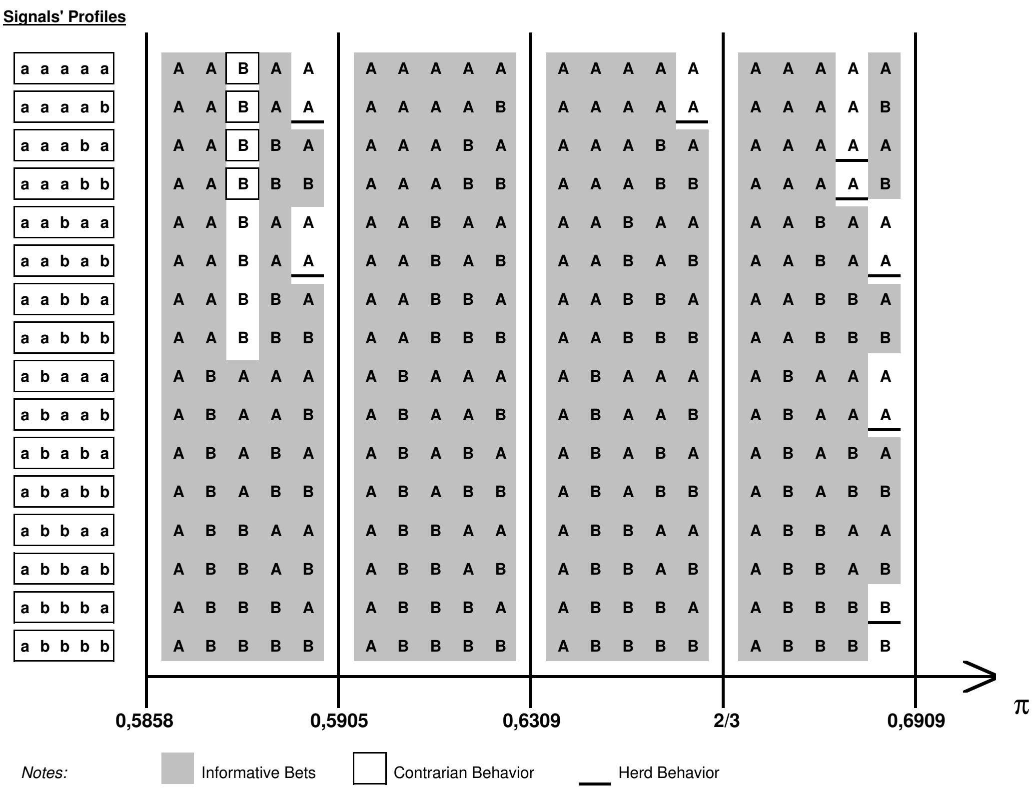

Figure 2: Set of Outcomes for the First Five Myopic Bettors with Middle Values of π.

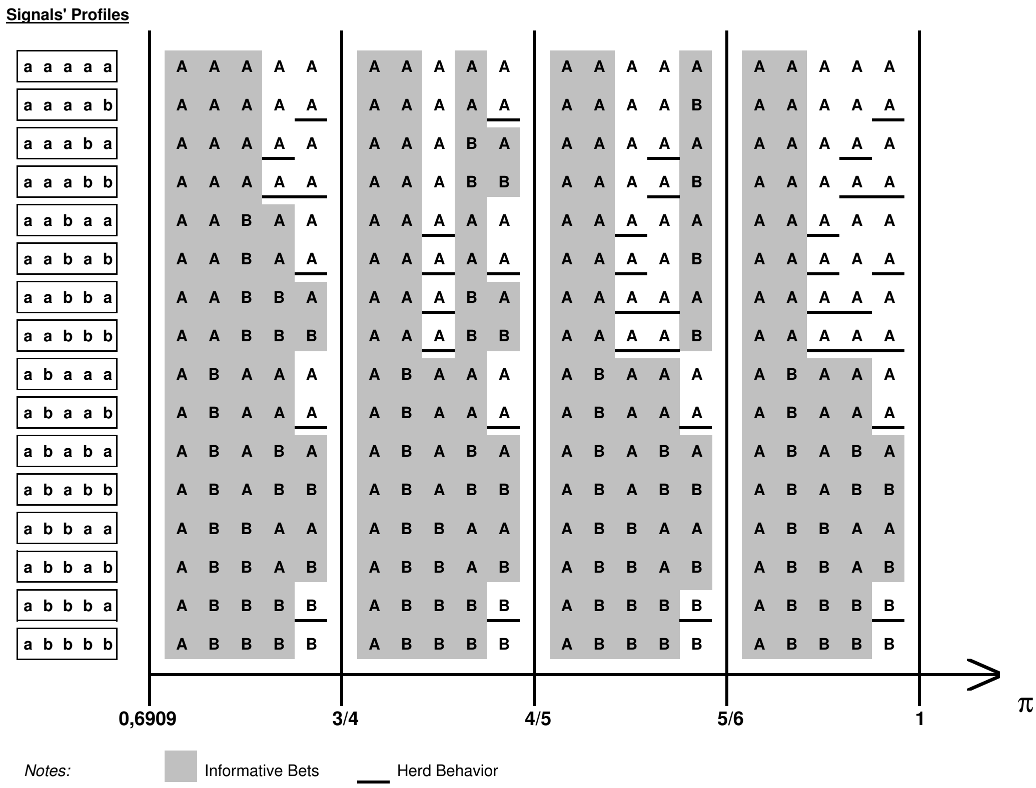

Figure 3: Set of Outcomes for the First Five Myopic Bettors with High Values of π.

Some Examples of Informational Inefficiencies

Since short-run herding and non-informative bets in general can arise, valuable information may be lost in the parimutuel betting market. In this respect, it seems worthwhile to investigate more deeply whether significant bandwagon effects in the betting odds can be created. More generally, we now examine if herd and contrarian behaviors (which are non-informative behaviors by definition) can lead to significant mispricing. To address this issue, we compare implicit prices set in the market—represented by horses' *subjective probabilities*—with objective probabilities.

The subjective probability of period i that horse H wins the race depends on the betting odds established in period i, and is given by

$$P_H^S(s^i) \equiv \frac{h(s^i)}{i} = \frac{1}{O_H(s^i)+1}.$$

This variable can be viewed as the price of obtaining a claim to one unit of money in the event that horse H wins the race, and thus it is also called the (equilibrium) *implicit price* for horse H in period i if s^i is an equilibrium history of bets. To simplify the notations, let $P_H^S(s(\sigma \mid q^i)) = P_H^S(q^i)$, where σ is an equilibrium strategy profile. The *objective probability* of period i that horse H wins the race is given by

$$P_H^O(q^i) \equiv \Pr(\theta_H \mid q^i),$$

where $q^i = (q_1, \ldots, q_i)$ is the vector of signals received by all bettors up to period i. Hence, $P_H^O(q^i)$ is the belief about the winning chance of horse H for an agent who would be able to observe the signals of all bettors up to period i. Said differently, if bettors before bettor i use the perfectly revealing strategy, then $\mu_i(\theta_H \mid s^{i-1}; q_i) = P_H^O(q^i)$.

By means of simulations, we now illustrate how inefficient can win pool shares (implicit prices) be in forecasting outcomes in a sequential parimutuel betting market. The initial conditions for the first simulation are the following: the quality of information is $\pi = 0.85$ and the profile of private signals is $q^{15} = (q^B, q^B, q^A, q^A, q^A, q^B, q^A, q^A, q^A, q^A, q^A, q^A, q^A, q^A, q^A)$. Figure 4 on the next page shows the evolution of bettors' beliefs about horse A, $\mu_i(\theta_A \mid s^{i-1}; q_i)$, the evolution of the implicit price for horse A, $P_A^S(s^i)$, and the evolution of the objective probability of horse A, $P_A^O(q^i)$, up to period 15. After fifteen periods of bets, the implicit price for horse A and bettors' beliefs about the winning chance of horse A are close to zero, whereas the objective probability of horse A is close to 1. Here, a sufficiently excess of bets on horse B at the beginning of the sequence leads bettors to believe that horse B is more likely to win, regardless of their own private information. Hence, informational efficiency of betting odds is severely disrupted due to long sequences of imitative bets.

The quality of information used in the first simulation is relatively extreme and one might believe that such inefficiencies cannot arise with intermediate qualities of information. However, this is not the case. Figure 5 on page 21 shows the 15-period moving of

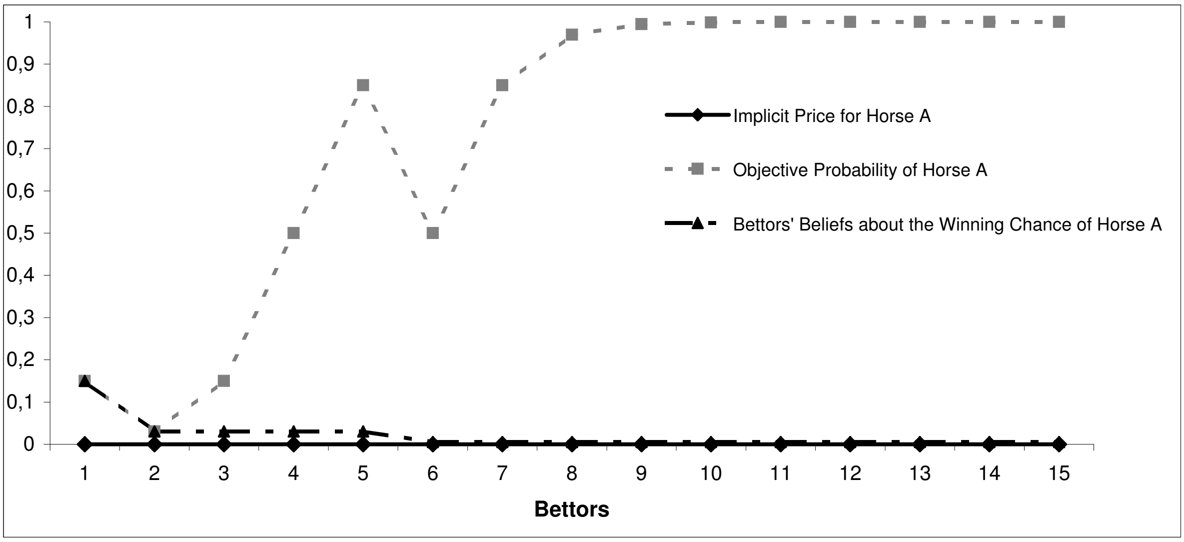

Figure 4: Evolution of Beliefs, Implicit Prices, and Objective Probabilities of Horse A up to Bettor 15 with $\pi = 0.85$ and $q^{15} = (q^B, q^B, q^A, q^A, q^A, q^B, q^A, q^A, q^A, q^A, q^A, q^A, q^A, q^A, q^A)$.

beliefs, implicit prices and objective probabilities of horse A with the sequence of private signals $q^{15} = (q^A, q^A, q^A, q^B, q^B, q^A, q^B, q^B, q^B, q^B, q^B, q^B, q^A, q^B, q^B)$ and with the quality of information $\pi = 0.7$. Here again, significant mispricing and discrepancies between beliefs and objective probabilities are observed. Heavy bets on a horse trigger more bets on that horse, even though this horse turned out not to be the one that will win the race according to the information which is distributed among all bettors. Nevertheless, this inefficiency is stronger to obtain with $\pi = 0.7$ than with $\pi = 0.85$ since three initial decisions on the "wrong" horse are necessary to generate herding. For example, as illustrated in Figure 6 on the next page, prices set in the market can converge to the fundamentals for other profiles of signals. Of course, implicit prices adjust relatively slowly after some periods because of the relative rigidity of betting odds.

Finally, it is worth noticing that significant mispricing is possible even if the first decisions correctly reflect the fundamentals. Indeed, as illustrated in Figure 7 on page 22, herding might arise later due to non-informative bets during the last five periods.

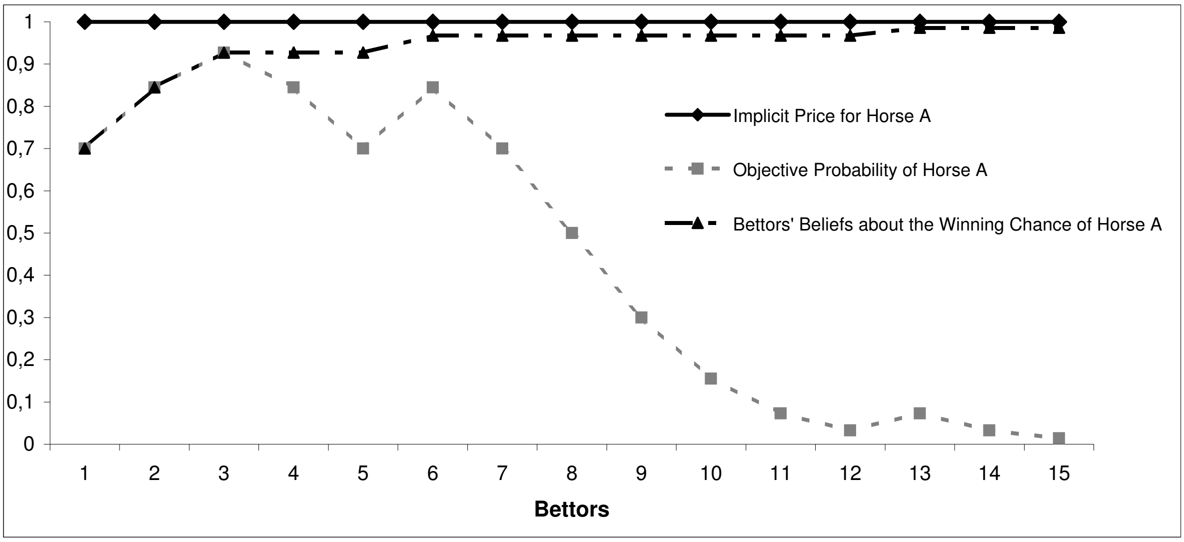

Figure 5: Evolution of Beliefs, Implicit Prices, and Objective Probabilities of Horse A up to Bettor 15 with $\pi = 0.70$ and $q^{15} = (q^A, q^A, q^A, q^B, q^B, q^A, q^B, q^B, q^B, q^B, q^B, q^B, q^A, q^B, q^B)$.

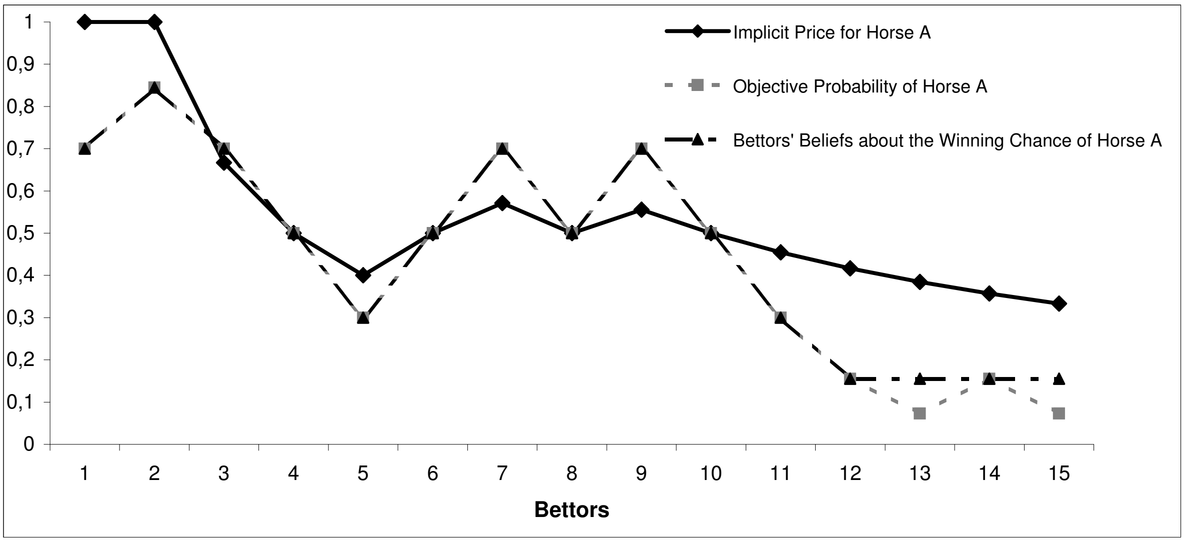

Figure 6: Evolution of Beliefs, Implicit Prices, and Objective Probabilities of Horse A up to Bettor 15 with $\pi = 0.70$ and $q^{15} = (q^A, q^A, q^B, q^B, q^B, q^A, q^A, q^B, q^A, q^B, q^B, q^B, q^B, q^A, q^B)$.

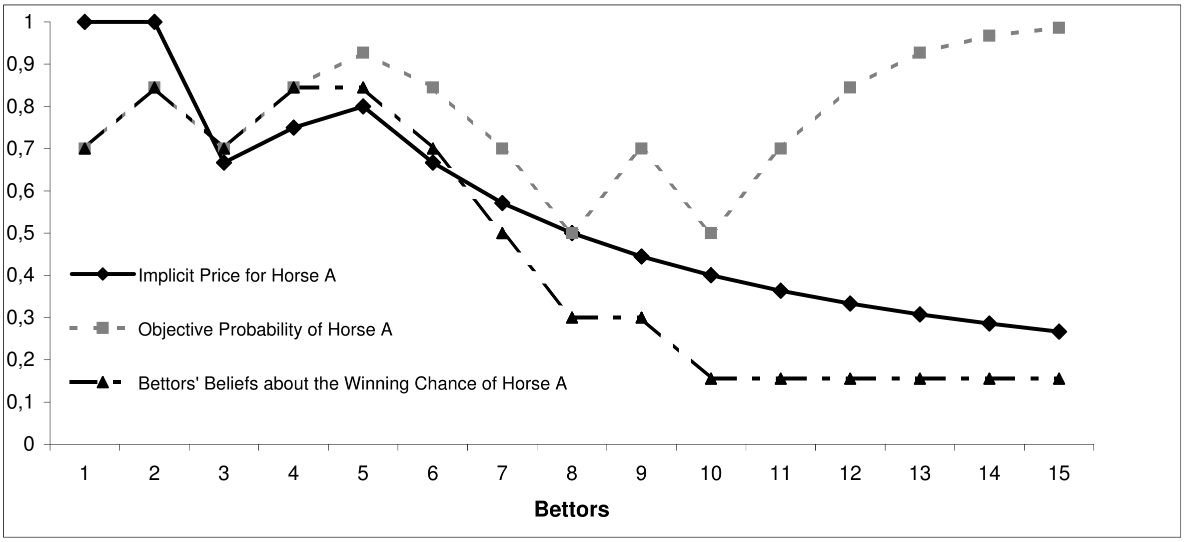

Figure 7: Evolution of Beliefs, Implicit Prices, and Objective Probabilities of Horse A up to Bettor 15 with $\pi = 0.70$ and $q^{15} = (q^A, q^A, q^B, q^A, q^A, q^B, q^B, q^B, q^A, q^B, q^A, q^A, q^A, q^A, q^A)$.

5 Conclusion and Relation to Previous Work

To the best of our knowledge, the analysis offered in this paper is the first to investigate betting behavior in parimutuel markets under asymmetric information. Not surprisingly, the temporal structure of the market drives the results. Though betting odds perfectly reflect all available information in the simultaneous move market game, non-informative behaviors emerge whatever the quality of players' information in a sequential parimutuel betting market. Those informational inefficiencies at various states of the world are notably due to herd behavior arising with intermediate and high qualities of information, leading to extreme mispricing effects in some identifiable states of the world. These results are qualitatively robust to the assumptions made concerning bettors' expectations (rational or myopic expectations).

In the following lines we first sketch a short bibliographic note on parimutuel betting games. Then we discuss the relationships which exist between our framework and the growing literature on social learning and information aggregation in sequential decision models with observable actions and asymmetric information.

Watanabe, Nonoyama, and Mori (1994) are the first to analyze a game-theoretical model of parimutuel betting. They consider two horses and assumed that bettors' prior beliefs are inconsistent (agents agree to disagree). An ad hoc selection criterion is introduced to select a unique equilibrium which is regular in the sense that if there exists a player who bets on horse A (respectively B) at an equilibrium, then players who have strictly larger (respectively smaller) probability for the win of this horse will always bet on it. Watanabe (1997) considers the same model but with a continuum of bettors. Feeney and King (2001) have developed a sequential betting game with consistent beliefs, a finite set of strategic bettors, and no withdrawing possibilities.[16] By assuming symmetric information (and common prior beliefs on the winning chance of each horse), they show that the subgame perfect equilibrium is unique for almost all values of prior probabilities. In such an equilibrium, there is always an advantage to being an early mover and players betting first do not necessarily choose to bet on the horse with the highest probability to win.[17] As shown by Propositions 6 and 7 of this paper, at least with two and three periods, asymmetric information has "destroyed" this effect according to which early bettors might choose an action with a low ex ante expected gain.[18]

Our results can also be linked to those obtained in traditional herding models, in models of sequential trade in financial markets, and in the literature on sequential voting. In particular, it seems natural to draw a parallel between the analysis offered here and the literature on "information cascades". An information cascade can be defined as a choice sequence in which some agents act as if they ignored their private information and followed the choices made

[16] A game-theoretical model under symmetric information where it is assumed that each bettor must spend his entire endowment on the race has also been analyzed by Chadha and Quandt (1996).

[17] This model was extended to the case where bettors have the possibility to withdraw by Koessler, Ziegelmeyer, and Broihanne (2003), by introducing a fraction of noise bettors into the game (see also Hurley and McDonough, 1995 and Terrell and Farmer, 1996 in simultaneous move betting games with transactions/information costs).

[18] Of course, this effect cannot arise with myopic bettors.

earlier in the sequence by other agents (see, e.g., Banerjee, 1992, Bikhchandani, Hirshleifer, and Welch, 1992, and Chamley and Gale, 1994). In the pioneering herding models agents are only concerned with maximizing their own expected payoffs which cannot be *directly* affected by the actions of others. The complete absence of payoff interdependencies is obviously a strong assumption. In this respect, our work can be seen as an extension of rational herd behavior models where negative payoff externalities are considered. Even though long-run herding is impossible in sequential parimutuel betting markets, our results imply that herd behavior, at least in the short-run, is robust to the parimutuel mechanism.

Though most of the literature on rational herding assumes that prices for taking an action are fixed, a notable exception is Avery and Zemsky's (1998) model where the relationship between asset prices and herd behavior is investigated. Hence, as in sequential parimutuel betting markets under asymmetric information, along with informational externalities, payoff externalities arise through the addition of a price mechanism. The main differences between Avery and Zemsky's (1998) asset pricing model and parimutuel betting markets are the following. First, in Avery and Zemsky's (1998) model, the price is determined by a market maker according to his information about past trades, while in parimutuel betting markets the price mechanism is exogenous and ensures that average bettors' return is null or negative (negative if transaction costs are strictly positive). Second, in parimutuel betting markets the return of each player also depends on his expectation about the behavior of later participants. Hence, in such markets, the analysis is complicated by the fact that fully rational agents are concerned not only with learning from predecessors, but also with signaling to successors.[19] This intractable difficulty was avoided in Subsection 4.4 by considering myopic agents. This latter framework is in fact very related to the one of Avery and Zemsky. More precisely, if we assume further that bettors do not recognize their impact on betting odds, i.e., if $u_i(s^i, \theta_H) = O_H(s^{i-1}) + 1$ if $s_i = H$ and $u_i(s^i, \theta_H) = 0$ if $s_i \neq H$, and if odds against each horse H is replaced by $O_H(s^{i-1}) = \frac{1-\Pr(\theta_H|s^{i-1})}{\Pr(\theta_H|s^{i-1})}$, then the subjective probability of horse H becomes equal to $\Pr(\theta_H \mid s^{i-1})$, i.e., the price of the asset in Avery and Zemsky (1998). In that case, condition (8) on page 15 for bettor i to follow his signal is replaced by $\mu_i(\theta_H \mid s^{i-1}; q_i) > \Pr(\theta_H \mid s^{i-1})$, which is satisfied iff $q_i = q^H$. Hence, bettors always follow their private signal. Our results contrast with Avery and Zemsky's (1998) ones since in Avery and Zemsky full revelation of information is obtained once an endogenous price is incorporated into the analysis. To obtain herd behavior, they have to consider information asymmetries between the market maker and traders. This is not necessary in parimutuel betting markets where the price mechanism is exogenous and independent of the types of participants. In particular, betting odds do not integrate bettors' decision rules (the price mechanism does not depend on the assumptions made concerning agents' behavior).

Finally, our work can be related to the literature on sequential voting in the sense that

[19]Dasgupta (2000) performs a similar analysis with the notable difference that he considers positive payoff externalities with an additional requirement of complete agreement on investment decisions. See also Corsetti, Dasgupta, Morris, and Shin (2004) in a 2-period model.

this literature also extends information cascade models to a context in which an individual's payoff from taking an action is influenced by the actions of others.[20] In particular, agents must consider the information their actions provide to agents later in the sequence. This literature also asks how sequential decisions differ from simultaneous decisions in terms of generated outcomes and information aggregation. However, betting markets are characterized by strong conflicting interests that we do not find in standard models of sequential voting. Additionally, only a player which is pivotal in voting models can affect the outcome, whereas all players always affect the payoffs of the others in parimutuel games. Our results, particularly the impossibility results concerning perfectly revealing equilibria, differ from those of this literature because our impossibility results do not require heterogeneous qualities of information or more than two horses. This contrasts with the results obtained in sequential voting where there always exists a perfectly revealing equilibrium when there is only one quality and two candidates (see Theorem 1 in Dekel and Piccione, 2000).

Appendix

Proof of Proposition 2. Assume that $k < n$ bettors always follow their signal, a bettors always bet on horse A, and $b = n - a - k$ bettors always bet on horse B. Let N_K be the set of bettors who follow their signals, N_A the set of bettors who always bet on horse A, and N_B the set of bettors who always bet on horse B. For each bettor $i \in N_K$ (in case $k \neq 0$) we have

$$U_i(A, \sigma_{-i} \mid q^A) = n\pi \left(\sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{a+1+j} \right)$$

$$U_i(B, \sigma_{-i} \mid q^A) = n(1-\pi) \left(\sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{b+1+j} \right)$$

$$U_i(A, \sigma_{-i} \mid q^B) = n(1-\pi) \left(\sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{a+1+j} \right)$$

$$U_i(B, \sigma_{-i} \mid q^B) = n\pi \left(\sum_{j=0}^{k-1} \frac{C_{k-1}^j \pi^j (1-\pi)^{k-1-j}}{b+1+j} \right).$$

[20]See, e.g., Feddersen and Pesendorfer (1998), Dekel and Piccione (2000), and references therein.

For each bettor $i \in N_A$ we have

$$U_i(A, \sigma_{-i} \mid q^A) = n\pi \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+j} \right)$$

$$U_i(B, \sigma_{-i} \mid q^A) = n(1-\pi) \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+1+j} \right)$$

$$U_i(A, \sigma_{-i} \mid q^B) = n(1-\pi) \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+j} \right)$$

$$U_i(B, \sigma_{-i} \mid q^B) = n\pi \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+1+j} \right).$$

Finally, for each bettor $i \in N_B$ we have

$$U_i(A, \sigma_{-i} \mid q^A) = n\pi \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+1+j} \right)$$

$$U_i(B, \sigma_{-i} \mid q^A) = n(1-\pi) \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+j} \right)$$

$$U_i(A, \sigma_{-i} \mid q^B) = n(1-\pi) \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+1+j} \right)$$

$$U_i(B, \sigma_{-i} \mid q^B) = n\pi \left(\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{b+j} \right).$$

It is easy to verify that each bettor $i \in N_A$ is rational only if $a < b+1$ and each bettor $i \in N_B$ is rational only if $b < a+1$, which implies that $b-1 < a < b+1$, i.e., $a = b = \frac{n-k}{2}$. In this case, it is rational for each bettor $i \in N_K$ to follow his signal. Moreover, each bettor $i \in N_H$ rationally follows his signal when he receives the signal q^H. It remains to check under which conditions each bettor $i \in N_H$ bets on H even when he receives the signal $q^{\overline{H}}$. From the expected utilities given above, this condition is satisfied if and only if

$$\frac{\pi}{1-\pi} \leq \frac{\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+j}}{\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{a+1+j}} = \frac{\sum_{j=0}^{k} \frac{(\frac{\pi}{1-\pi})^j}{j!(k-j)!(a+j)}}{\sum_{j=0}^{k} \frac{(\frac{\pi}{1-\pi})^j}{j!(k-j)!(a+j+1)}},$$

where $a = \frac{n-k}{2}$ and $0 \leq k < n$. The fact that there are exactly $C_n^k \times C_{n-k}^{\frac{n-k}{2}}$ partially informative equilibria with k informative bets is obvious. □

Proof of Proposition 3. To show that values of π satisfying the condition for a k-informative equilibrium to exist belong to an interval $[1/2, \pi(k,n)]$, where $\pi(k,n) < 1$, we begin to give in the following lemma a simpler formulation of Inequality (3). Then, we present in the next two lemmas useful properties of this new formulation that are sufficient to prove the proposition.

Lemma 5 *Inequality (3) is satisfied if and only if $g(k,n,\pi) \leq 0$, where*

$$g(k,n,\pi) = \sum_{j=0}^{k+1} \left(\frac{\pi}{1-\pi}\right)^j C_{k+1}^j \frac{k-2j+1}{k-2j-n}.$$

Proof. Let $f(k,n,\pi) = \frac{\pi}{1-\pi} - \frac{\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+j}}{\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+1+j}}$. We have to show that $f(k,n,\pi) \leq 0$ is equivalent to $g(k,n,\pi) \leq 0$. Let $x = \frac{\pi}{1-\pi} \in]1,+\infty[$ and $a = \frac{n-k}{2}$. We have $f(k,n,\pi) \leq 0 \Leftrightarrow x \leq \frac{\sum_{j=0}^{k} \frac{C_k^j x^j}{a+j}}{\sum_{j=0}^{k} \frac{C_k^j x^j}{a+j+1}} \Leftrightarrow \sum_{j=-1}^{k} \frac{C_k^j x^{j+1}}{a+j+1} \leq \sum_{j=0}^{k+1} \frac{C_k^j x^j}{a+j}$, since by convention $C_k^{-1} = C_k^{k+1} = 0$. The last inequality is equivalent to $\sum_{j=0}^{k+1} \frac{x^j}{a+j}(C_k^j - C_k^{j-1}) \geq 0$. Using the definition of the binomial coefficient, it is not difficult to show that $C_k^j - C_k^{j-1} = C_{k+1}^j \frac{k+1-2j}{k+1}$. Substituting this value into the last inequality, we get $\sum_{j=0}^{k+1} \frac{x^j}{a+j} C_{k+1}^j \frac{k+1-2j}{k+1} \geq 0 \Leftrightarrow \sum_{j=0}^{k+1} x^j C_{k+1}^j \frac{k+1-2j}{(\frac{n-k}{2}+j)(k+1)} \geq 0 \Leftrightarrow \sum_{j=0}^{k+1} x^j C_{k+1}^j \frac{k+1-2j}{n-k+2j} \geq 0 \Leftrightarrow g(k,n,\pi) \leq 0$. □

Lemma 6 *For all $n \geq 2$ and $k < n$, there exists π, $\pi' \in]1/2,1[$ such that $g(k,n,\pi) < 0$ and $g(k,n,\pi') > 0$.*

Proof. It is equivalent to prove this property for $f(k,n,\pi)$. We have

$$f(k,n,1/2) = 1 - \frac{\sum_{j=0}^{k} \frac{1}{j!(k-j)!(\frac{n-k}{2}+j)}}{\sum_{j=0}^{k} \frac{1}{j!(k-j)!(\frac{n-k}{2}+j+1)}} < 0.$$

Moreover, $\lim_{\pi \to 1^-} \frac{\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+j}}{\sum_{j=0}^{k} \frac{C_k^j \pi^j (1-\pi)^{k-j}}{\frac{n-k}{2}+1+j}} = \frac{\frac{n-k}{2}+k+1}{\frac{n-k}{2}+k}$, and $\lim_{\pi \to 1^-} \frac{\pi}{1-\pi} = +\infty$, which implies that $\lim_{\pi \to 1^-} f(k,n,\pi) > 0$ for all n and k. Hence, there exists π, $\pi' \in]1/2,1[$ such that $f(k,n,\pi) < 0$ and $f(k,n,\pi') > 0$. □

Lemma 7 *For all $n \geq 2$ and $k < n$, $g(k,n,\cdot)$ is strictly increasing with π on the interval $]1/2,1[$.*

Proof. We have to show that $\frac{\partial g(k,n,\pi)}{\partial \pi} > 0$ for all $\pi \in]1/2,1[$. Let $x = \frac{\pi}{1-\pi} \in]1,+\infty[$. We have $\frac{\partial g(k,n,\pi)}{\partial \pi} > 0 \Leftrightarrow \sum_{j=0}^{k+1} j\, x^{j-1} C_{k+1}^j \frac{k-2j+1}{k-2j-n} > 0$. Assume that k is an even number. Then, the last inequality is equivalent to

$$\begin{aligned}
& \sum_{j=0}^{k/2} j\, x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=(k/2)+1}^{k+1} j\, x^{j-1} C_{k+1}^j \frac{2j-k-1}{n+2j-k} \\
\Leftrightarrow \quad & \sum_{j=0}^{k/2} j\, x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=0}^{k/2} (k+1-j)\, x^{k-j} C_{k+1}^{k+1-j} \frac{2(k+1-j)-k-1}{n+2(k+1-j)-k} \\
\Leftrightarrow \quad & \sum_{j=0}^{k/2} j\, x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < \sum_{j=0}^{k/2} (k+1-j)\, x^{k-j} C_{k+1}^j \frac{k-2j+1}{n-2j+k+2},
\end{aligned}$$

because $C_{k+1}^j = C_{k+1}^{k+1-j}$. To prove the last inequality, it is sufficient to show that for all $j = 0,1,\ldots,k/2$ we have $j\, x^{j-1} C_{k+1}^j \frac{k-2j+1}{n+2j-k} < (k+1-j)\, x^{k-j} C_{k+1}^j \frac{k-2j+1}{n-2j+k+2}$. Since $x > 1 \Rightarrow x^{j-1} < x^{k-j}$ for all $j = 0,1,\ldots,k/2$, it is sufficient to show that $j \frac{k-2j+1}{n+2j-k} \leq (k+1-j) \frac{k-2j+1}{n-2j+k+2}$ or, equivalently, $k+1-2j \geq 0$, which is satisfied for all $j = 0,1,\ldots,k/2$. Now, we assume that k is an odd number.

Then, we have

$$\begin{aligned}
&\sum_{j=0}^{(k-1)/2} j\,x^{j-1}C_{k+1}^{j}\frac{k-2j+1}{n+2j-k} < \sum_{j=(k+1)/2}^{k+1} j\,x^{j-1}C_{k+1}^{j}\frac{2j-k-1}{n+2j-k}\\
\Leftrightarrow\;&\sum_{j=0}^{(k-1)/2} j\,x^{j-1}C_{k+1}^{j}\frac{k-2j+1}{n+2j-k} < \sum_{j=(k+3)/2}^{k+1} j\,x^{j-1}C_{k+1}^{j}\frac{2j-k-1}{n+2j-k}\\
\Leftrightarrow\;&\sum_{j=0}^{(k-1)/2} j\,x^{j-1}C_{k+1}^{j}\frac{k-2j+1}{n+2j-k} < \sum_{j=0}^{(k-1)/2} (k+1-j)\,x^{k-j}C_{k+1}^{k+1-j}\frac{k-2j+1}{n-2j+k+2}.
\end{aligned}$$

The rest of the proof is as before. □

Proof of Proposition 4. We show that when the number of bettors tend to infinity the equilibrium condition for a bettor who does not use the fully informative strategy is not satisfied. Consider a sequence of k_n-equilibria, $n \geq 2$, and assume by way of contradiction that there is an infinite subsequence $(k_n)_n$ with $k_n > 0$ for all n, i.e., an infinite subsequence of equilibria that are not separating. Consider without loss of generality a player who always bets on A. For this player, denoted by i, the expected payoff for choosing A must be higher than the expected payoff for choosing B even when this player has the signal $q_i = q^B : U_i(A, \sigma_{-i} \mid q^B) \geq U_i(B, \sigma_{-i} \mid q^A)$. By Proposition 2 we know that the number of bettors who always bet on A is the same as the number of bettors who always bet on B, i.e., it is equal to $\frac{n-k_n}{2}$. Hence, the last inequality is equivalent to

$$(1-\pi)\,\mathrm{E}\left[\frac{n}{\frac{n-k_n}{2} + \sum_{j=1}^{k_n} \mathbf{1}_{[q_j=q^A]}} \;\Big|\; \theta_A\right] \geq \pi\,\mathrm{E}\left[\frac{n}{1+\frac{n-k_n}{2} + \sum_{j=1}^{k_n} \mathbf{1}_{[q_j=q^B]}} \;\Big|\; \theta_B\right] \tag{10}$$

If $(k_n)_n$ is bounded, then when $n \to \infty$ this condition becomes $2(1-\pi) \geq 2\pi$, a contradiction. If $(k_n)_n$ is not bounded, then we can extract a strictly increasing subsequence, for which condition (10) becomes, for $n \to \infty$,

$$(1-\pi)\frac{n}{\frac{n-k_n}{2} + \pi\,k_n} \geq \pi\frac{n}{1+\frac{n-k_n}{2} + \pi\,k_n} \;\Leftrightarrow\; (1-\pi)\frac{2n}{n+k_n(2\pi-1)} \geq \pi\frac{2n}{2+n+k_n(2\pi-1)},$$

which also yields to the contradiction $1-\pi > \pi$ when $n \to \infty$. Therefore, for n sufficiently large we know that there is no k_n-informative equilibrium with $k_n > 0$, which proves the proposition. □

Proof of Proposition 6. To prove the proposition we analyze all possible strategies of the first bettor and we examine the associated best response of the second one depending on his possible outside equilibrium beliefs. The first bettor has three types of possible strategies: i) He follows his signal; ii) he bets against his signal; iii) he bets non-informatively.

i) *The First Bettor Follows his Signal.* We check for the existence of an equilibrium in which the first bettor always follows his signal, i.e., $\sigma_1(q^A) = A$ and $\sigma_1(q^B) = B$. In this case, Bayes' rule applies everywhere and we obtain the following beliefs for the second bettor: $\mu_2(\theta_A \mid A; q^A) = \mu_2(\theta_B \mid B; q^B) = \frac{\pi^2}{\pi^2+(1-\pi)^2}$, $\mu_2(\theta_A \mid A; q^B) = \mu_2(\theta_B \mid B; q^A) = \mu_2(\theta_A \mid B; q^A) = \mu_2(\theta_B \mid A; q^B) = 1/2$, and $\mu_2(\theta_A \mid B; q^B) = \mu_2(\theta_B \mid A; q^A) = \frac{(1-\pi)^2}{\pi^2+(1-\pi)^2}$. From those beliefs, we can compute the expected utilities of bettor 2 depending on his signal, the observed history, and his betting choice. We easily obtain $U_2(A \mid A; q^A) = U_2(B \mid B; q^B) = \frac{\pi^2}{\pi^2+(1-\pi)^2}$, $U_2(A \mid B; q^B) = U_2(B \mid A; q^A) = \frac{2(1-\pi)^2}{\pi^2+(1-\pi)^2}$, $U_2(A \mid A; q^B) = U_2(B \mid B; q^A) = \frac{1}{2}$, and $U_2(A \mid B; q^A) = U_2(B \mid A; q^B) = 1$. Therefore, for any $H \in \{A, B\}$, $\sigma_2(\overline{H}; q^H) = H$ and

$$\sigma_2(H; q^H) = \begin{cases} H & \text{if } \pi > \frac{\sqrt{2}}{1+\sqrt{2}} \\ \overline{H} & \text{if } \pi < \frac{\sqrt{2}}{1+\sqrt{2}}. \end{cases}$$

It remains to verify if bettor 1 is rational for all $\pi > 1/2$ when bettor 2 uses the above strategy.

Given σ_2, we obtain the following expected utilities:

$$\begin{aligned}U_1(A,\sigma_2 \mid q^A) &= \mu_1(\theta_A \mid q^A) \sum_{q_2\in Q_2} \Pr(q_2 \mid \theta_A)(1+O_A(A,\sigma_2(A;q_2)))\\ &= \pi \sum_{q_2\in Q_2} \Pr(q_2 \mid \theta_A)\frac{2}{a(A,\sigma_2(A;q_2))}\\ &= \begin{cases}\pi(2-\pi) & \text{if } \pi > \frac{\sqrt{2}}{1+\sqrt{2}}\\ 2\pi & \text{otherwise.}\end{cases}\end{aligned}$$

Similarly,

$$\begin{aligned}U_1(A,\sigma_2 \mid q^A) = U_1(B,\sigma_2 \mid q^B) &= \begin{cases}\pi(2-\pi) & \text{if } \pi > \frac{\sqrt{2}}{1+\sqrt{2}}\\ 2\pi & \text{otherwise}\end{cases}\\ U_1(B,\sigma_2 \mid q^A) = U_1(A,\sigma_2 \mid q^B) &= \begin{cases}(1-\pi)(2-\pi) & \text{if } \pi > \frac{\sqrt{2}}{1+\sqrt{2}}\\ 2(1-\pi) & \text{otherwise.}\end{cases}\end{aligned}$$

We get $\sigma_1(q^H) = H$, for any $\pi > 1/2$. Hence, player 1 follows his own private signal.

ii) *The First Bettor Bets Against his Signal.* Assume that $\sigma_2(q^H) = \overline{H}$. Bayes' rule applies everywhere and we obtain the following beliefs for bettor 2: $\mu_2(\theta_H \mid \overline{H}; q^H) = \frac{\pi^2}{\pi^2+(1-\pi)^2}$, $\mu_2(\theta_H \mid H; q^H) = 1/2$, and $\mu_2(\theta_H \mid H; q^{\overline{H}}) = \frac{(1-\pi)^2}{\pi^2+(1-\pi)^2}$. Bettor 2's expected utility is then $U_2(H \mid H; q^{\overline{H}}) = \frac{(1-\pi)^2}{\pi^2+(1-\pi)^2}$, $U_2(H \mid H; q^H) = \frac{1}{2}$, $U_2(H \mid \overline{H}; q^{\overline{H}}) = 1$, and $U_2(H \mid \overline{H}; q^H) = \frac{2\pi^2}{\pi^2+(1-\pi)^2}$. It follows immediately that the second bettor always adopts a contrarian behavior, whatever his private signal: $\sigma_2(H, q_2) = \overline{H}$ for all $q_2 \in Q_2$. Now consider bettor 1. It is easy to see that $U_1(H \mid q^H) = 2\pi$ and $U_1(\overline{H} \mid q^H) = 2(1-\pi) < 2\pi$. Indeed, since the second bettor always bets on the other horse, the odds against the horse chosen by the first bettor is always equal to 2. Hence, there is no equilibrium where the first bettor always bets against his private signal.

iii) *The First Bettor Bets Non-Informatively.* Assume that $\sigma_1(q_1) = H \in \{A, B\}$ for all $q_1 \in Q_1$. The second bettor cannot apply Bayes' rule when he observes that the first bettor has bet on horse $\overline{H}$. When he observes H, he keeps his posterior beliefs. We obtain $\sigma_2(H; q^{\overline{H}}) = \overline{H}$ and

$$\sigma_2(H; q^H) = \begin{cases} H & \text{if } \pi > 2/3\\ \overline{H} & \text{if } \pi < 2/3\end{cases}$$

To determine bettor 2's optimal action when he has observed a first bet on horse $\overline{H}$ we have to specify outside equilibrium beliefs. By restricting our attention to consistent beliefs (in the sense of Kreps and Wilson, 1982), the following conditions must be satisfied: $\mu_2(\theta_H \mid \overline{H}; q^H) \in [1/2, \frac{\pi^2}{\pi^2+(1-\pi)^2}]$, and $\mu_2(\theta_H \mid \overline{H}; q^{\overline{H}}) \in [\frac{(1-\pi)^2}{\pi^2+(1-\pi)^2}, 1/2]$. Hence, $U_2(H \mid \overline{H}; q^H) = 2\mu_2(\theta_H \mid \overline{H}; q^H) \geq 1$ and $U_2(\overline{H} \mid \overline{H}; q^H) = 1 - \mu_2(\theta_H \mid \overline{H}; q^H) \leq 1/2$, which implies that $\sigma_2(\overline{H}; q^H) = H$. We obtain the following expected utilities for bettor 1:

$$\begin{aligned}U_1(H \mid \overline{H}) &= 2(1-\pi)(\pi/2 + (1-\pi)) = (1-\pi)(2-\pi) && \text{if } \pi > 2/3\\ U_1(H \mid \overline{H}) &= 2(1-\pi)(\pi + (1-\pi)) = 2(1-\pi) && \text{if } \pi < 2/3,\end{aligned}$$

and

$$\begin{aligned}U_1(\overline{H} \mid \overline{H}) &= 2\pi(\pi/2 + (1-\pi)) = \pi(2-\pi) && \text{if } \sigma_2(\overline{H}; q^{\overline{H}}) = \overline{H}\\ U_1(\overline{H} \mid \overline{H}) &= 2\pi(\pi + (1-\pi)) = 2\pi && \text{if } \sigma_2(\overline{H}; q^{\overline{H}}) = H.\end{aligned}$$

It is clear that if $\pi > 2/3$ or $\sigma_2(\overline{H}; q^{\overline{H}}) = H$, then bettor 1 deviates by following his signal $q^{\overline{H}}$. Hence, assume that $\pi < 2/3$ and $\sigma_2(\overline{H}; q^{\overline{H}}) = \overline{H}$. This last strategy is used by bettor 2 if $U_2(\overline{H} \mid \overline{H}; q^{\overline{H}}) \geq U_2(H \mid \overline{H}; q^{\overline{H}})$. Since $U_2(\overline{H} \mid \overline{H}; q^{\overline{H}}) \leq \frac{\pi^2}{\pi^2+(1-\pi)^2}$ and $U_2(H \mid \overline{H}; q^{\overline{H}}) \geq \frac{2(1-\pi)^2}{\pi^2+(1-\pi)^2}$, we must have $\pi \geq \frac{\sqrt{2}}{1+\sqrt{2}}$. Thereafter, note that $U_1(H \mid \overline{H}) \geq U_1(\overline{H} \mid \overline{H})$ if and only if $\pi \leq \frac{\sqrt{2}}{1+\sqrt{2}}$. Thus, whenever $\pi \neq \frac{\sqrt{2}}{1+\sqrt{2}}$, the first bettor never uses a non-informative strategy. □

Proof of Proposition 7. First note that if the first and the second bettors follow their signals, then there is no outside equilibrium beliefs to specify. Beliefs of the second bettor are the same as in the 2-bettor case. Beliefs of the third bettor are given by $\mu_3(\theta_A \mid A, A; q^A) = \frac{\pi^3}{\pi^3+(1-\pi)^3}$, $\mu_3(\theta_A \mid B, B; q^B) = \frac{(1-\pi)^3}{\pi^3+(1-\pi)^3}$, $\mu_3(\theta_A \mid A, A; q^B) = \mu_3(\theta_A \mid A, B; q^A) = \mu_3(\theta_A \mid B, A; q^A) = \pi$, $\mu_3(\theta_A \mid B, B; q^A) = \mu_3(\theta_A \mid B, A; q^B) = \mu_3(\theta_A \mid A, B; q^B) = 1 - \pi$. From those beliefs we can compute the expected utilities of the third bettor, and then his optimal strategy: $\sigma_3(H, \overline{H}; q^H) = \sigma_3(\overline{H}, H; q^H) = H$, for all π, and

$$\sigma_3(H, H; q^H) = \begin{cases} H & \text{if } \pi > \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \\ \overline{H} & \text{if } \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \end{cases}$$

$$\sigma_3(H, H; q^{\overline{H}}) = \begin{cases} H & \text{if } \pi > \frac{3}{4} \\ \overline{H} & \text{if } \pi < \frac{3}{4}. \end{cases}$$

Thereafter, we get the expected utilities of the second bettor:

$$U_2(A, \sigma_3 \mid A; q^A) = U_2(B, \sigma_3 \mid B; q^B) = \begin{cases} \frac{\pi^2}{\pi^2+(1-\pi)^2} & \text{if } \pi > 3/4 \\ \frac{\pi^2}{\pi^2+(1-\pi)^2}\frac{3-\pi}{2} & \text{if } 3/4 > \pi > \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \\ \frac{\pi^2}{\pi^2+(1-\pi)^2}\frac{3}{2} & \text{if } \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \end{cases}$$

$$U_2(B, \sigma_3 \mid A; q^A) = U_2(A, \sigma_3 \mid B; q^B) = \frac{(1-\pi)^2}{\pi^2+(1-\pi)^2}\frac{3(2-\pi)}{2} \quad \text{for all } \pi$$

$$U_2(A, \sigma_3 \mid A; q^B) = U_2(B, \sigma_3 \mid B; q^A) = \begin{cases} \frac{1}{2} & \text{if } \pi > 3/4 \\ \frac{3-\pi}{4} & \text{if } 3/4 > \pi > \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \\ \frac{3}{4} & \text{if } \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \end{cases}$$

$$U_2(A, \sigma_3 \mid B; q^A) = U_2(B, \sigma_3 \mid A; q^B) = \frac{3(2-\pi)}{4} \quad \text{for all } \pi.$$

We now turn to the optimal strategy of the second bettor. From the previous expected utilities it is clear that $U_2(H, \sigma_3 \mid \overline{H}; q^H) \geq U_2(\overline{H}, \sigma_3 \mid \overline{H}; q^H)$ for all $H \in \{A, B\}$ and $\pi \geq 1/2$. Hence, if the private signal of the second bettor contradicts the private signal of the first one, then the second bettor always follows his private signal (and thus bets against the first bettor's choice). Otherwise, we have:

If $\pi \geq 3/4$, then $U_2(A, \sigma_3 \mid A; q^A) \geq U_2(B, \sigma_3 \mid A; q^A) \Leftrightarrow \pi^2 \geq (3/2)(1-\pi)^2(2-\pi) \Leftrightarrow$

$$\pi \geq \frac{1}{9}\left(10 - \frac{35}{(27\sqrt{179}-296)^{1/3}} + (27\sqrt{179}-296)^{1/3}\right) \simeq 0.592 < 3/4.$$

If $3/4 \geq \pi \geq \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}$, then $U_2(A, \sigma_3 \mid A; q^A) \geq U_2(B, \sigma_3 \mid A; q^A) \Leftrightarrow \pi^2(3-\pi) \geq 3(1-\pi)^2(2-\pi) \Leftrightarrow$

$$\pi \geq \frac{1}{2}\left(3 - \frac{1}{(\sqrt{37}-6)^{1/3}} + (\sqrt{37}-6)^{1/3}\right) \simeq 0.571 < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}.$$

If $\pi \leq \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}$, then $U_2(A, \sigma_3 \mid A; q^A) \geq U_2(B, \sigma_3 \mid A; q^A) \Leftrightarrow \pi^2 \geq (1-\pi)^2(2-\pi) \Leftrightarrow$

$$\pi \geq 1 - 2\left(\frac{2}{3(\sqrt{177}-9)}\right)^{1/3} + \frac{(\frac{1}{2}(\sqrt{177}-9))^{1/3}}{3^{2/3}} \simeq 0.547 < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}.$$

Thus, bettor 2 always follows his signal if $\pi \geq 1 - 2\left(\frac{2}{3(\sqrt{177}-9)}\right)^{1/3} + \frac{(\frac{1}{2}(\sqrt{177}-9))^{1/3}}{3^{2/3}} \simeq 0.547$.

Otherwise, he goes against the trend.

We assume in the following lines that $\pi \geq 0.547$ (otherwise, the equilibrium we are currently analyzing breaks down). Recalling that we assume that the second bettor always follows his own private signal and that the third bettor acts according to his strategy σ_3 found before (which depends on π), we get (after some calculations) the following expected utilities of the first bettor:

$U_1(A, \sigma_2, \sigma_3 \mid q^A) = U_1(B, \sigma_2, \sigma_3 \mid q^B) =$

$$\begin{cases} \frac{\pi}{2}(2\pi^2 + 5\pi(1-\pi) + 6(1-\pi)^2) & \text{if } \pi > 3/4 \\ \pi(\pi^2 + 3\pi(1-\pi) + 3(1-\pi)^2) & \text{if } 3/4 > \pi > \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \\ \frac{3\pi}{2}(\pi^2 + 2\pi(1-\pi) + 2(1-\pi)^2) & \text{if } \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}, \end{cases}$$

$U_1(B, \sigma_2, \sigma_3 \mid q^A) = U_1(A, \sigma_2, \sigma_3 \mid q^B) =$

$$\begin{cases} \frac{1-\pi}{2}(6\pi^2 + 5\pi(1-\pi) + 2(1-\pi)^2) & \text{if } \pi > 3/4 \\ (1-\pi)(3\pi^2 + 3\pi(1-\pi) + (1-\pi)^2) & \text{if } 3/4 > \pi > \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}} \\ \frac{3(1-\pi)}{2}(2\pi^2 + 2\pi(1-\pi) + (1-\pi)^2) & \text{if } \pi < \frac{\sqrt[3]{3}}{1+\sqrt[3]{3}}. \end{cases}$$

We have exactly $U_1(H, \sigma_2, \sigma_3 \mid q^H) \geq U_1(\overline{H}, \sigma_2, \sigma_3 \mid q^H)$ if and only if $\pi \geq 1/2$. Hence, the first bettor always follows his own private signal. This completes the proof. □

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