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A universal inequality between the angular momentum and horizon area for axisymmetric and stationary black holes with surrounding matter

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Online at stacks.iop.org/CQG/25/162002**Abstract**

We prove that for sub-extremal axisymmetric and stationary black holes with arbitrary surrounding matter the inequality $8\pi|J| < A$ holds, where J is the angular momentum and A the horizon area of the black hole.

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1. Introduction

A well-known property of the Kerr solution, describing a single rotating black hole in vacuum, is given by

$$|p_J| \leq 1 \quad \text{with} \quad p_J := \frac{8\pi J}{A}, \quad (1)$$

where J and A denote the angular momentum and the horizon area of the black hole respectively. Equality in (1) holds if and only if the Kerr black hole is *extreme*. As was shown in [1], the equation $|p_J| = 1$ is even true more generally for axisymmetric and stationary black holes with surrounding matter in the degenerate limit (i.e. for vanishing surface gravity κ). Moreover, it was also conjectured in [1] that $|p_J| \leq 1$ still holds if the black hole is surrounded by matter. In this paper we prove this conjecture¹.

We start by requiring that a physically relevant non-degenerate black hole be characterized through the existence of trapped surfaces (i.e. surfaces with a negative expansion $\theta_{(l)}$ of outgoing null geodesics) in every sufficiently small interior vicinity of the event horizon. That is, in the terminology of [3], we concentrate on *sub-extremal* black holes. In the following we show that such surfaces cannot exist for $|p_J| \geq 1$, provided that an appropriate functional I

¹ Note that in [1] a more general conjecture, incorporating the black hole's electric charge Q , was formulated. Here we prove this conjecture for the pure Einstein field, i.e. for $Q = 0$, and vanishing cosmological constant $\Lambda = 0$. (It should be noted that, for $\Lambda \neq 0$, the inequality $|p_J| \leq 1$ can be violated. An example is the Kerr-(A)dS family of black holes, see [3].)

(to be defined below) cannot fall below 1. In turn, this can be proved by means of methods from the calculus of variations.

2. Coordinate systems and Einstein equations

We introduce suitable coordinates and metric functions by adopting our notation from [1], which is based on [2]. In spherical coordinates (r, θ, φ, t) , an exterior vacuum vicinity of the event horizon² can be described by the line element

$$ds^2 = e^{2\mu}(dr^2 + r^2 d\theta^2) + r^2 B^2 e^{-2\nu} \sin^2 \theta (d\varphi - \omega dt)^2 - e^{2\nu} dt^2. \quad (2)$$

The function B must be a solution of $\nabla \cdot (r \sin \theta \nabla B) = 0$, with $Br \sin \theta = 0$ at the event horizon \mathcal{H} . Here ∇ is the nabla operator in a three-dimensional flat space. In order to fix the coordinates we chose the particular solution $B = 1 - r_h^2/r^2$, where $r_h = \text{constant} > 0$. In this manner, we obtain coordinates in which \mathcal{H} is located at the coordinate sphere $r = r_h$. We now decompose the potential ν in the form $\nu = u + \ln B$, thereby obtaining three regular metric functions μ, u and ω , depending on r and θ only.

Following Bardeen [2], we introduce the new metric potentials $\hat{\mu} = r^2 e^{2\mu}$, $\hat{u} = r^2 e^{-2u}$, which are positive and regular functions of r and $\cos \theta$, as well as the new radial coordinate $R = \frac{1}{2}(r + \frac{r_h^2}{r})$. We arrive at the Boyer–Lindquist-type line element

$$ds^2 = \hat{\mu} \left(\frac{dR^2}{R^2 - r_h^2} + d\theta^2 \right) + \hat{u} \sin^2 \theta (d\varphi - \omega dt)^2 - \frac{4}{\hat{u}} (R^2 - r_h^2) dt^2, \quad (3)$$

which is singular at \mathcal{H} ($R = r_h$).

In these coordinates, the vacuum Einstein equations read as follows³:

$$(R^2 - r_h^2) \tilde{u}_{,RR} + 2R \tilde{u}_{,R} + \tilde{u}_{,\theta\theta} + \tilde{u}_{,\theta} \cot \theta = 1 - \frac{\hat{u}^2}{8} \sin^2 \theta \left(\omega_{,R}^2 + \frac{\omega_{,\theta}^2}{R^2 - r_h^2} \right), \quad (4)$$

$$(R^2 - r_h^2) (\omega_{,RR} + 4\omega_{,R} \tilde{u}_{,R}) + \omega_{,\theta\theta} + \omega_{,\theta} (3 \cot \theta + 4\tilde{u}_{,\theta}) = 0, \quad (5)$$

$$(R^2 - r_h^2) \tilde{\mu}_{,RR} + R \tilde{\mu}_{,R} + \tilde{\mu}_{,\theta\theta} = \frac{\hat{u}^2}{16} \sin^2 \theta \left(\omega_{,R}^2 + \frac{\omega_{,\theta}^2}{R^2 - r_h^2} \right) + R \tilde{u}_{,R} - (R^2 - r_h^2) \tilde{u}_{,R}^2 - \tilde{u}_{,\theta} (\tilde{u}_{,\theta} + \cot \theta), \quad (6)$$

where $\tilde{u} := \frac{1}{2} \ln (r_h^{-2} \hat{u})$ and $\tilde{\mu} := \frac{1}{2} \ln (r_h^{-2} \hat{\mu})$. At the horizon, the metric potentials obey the boundary conditions [2]

$$\mathcal{H}: \quad \omega = \text{constant} = \omega_h, \quad \frac{2r_h}{\sqrt{\hat{\mu}\hat{u}}} = \text{constant} = \kappa, \quad (7)$$

with the horizon angular velocity ω_h and the surface gravity κ . On the horizon's north and south pole ($R = r_h$ and $\sin \theta = 0$), the following regularity conditions hold:

$$\hat{\mu}(R = r_h, \theta = 0) = \hat{u}(R = r_h, \theta = 0) = \hat{\mu}(R = r_h, \theta = \pi) = \hat{u}(R = r_h, \theta = \pi) = \frac{2r_h}{\kappa}. \quad (8)$$

² For a stationary spacetime, the immediate vicinity of a black hole event horizon must be vacuum, see e.g. [2].

³ Throughout this paper we consider pure gravity, i.e. no electromagnetic fields, as well as vanishing cosmological constant, $\Lambda = 0$.

3. Necessary condition for the existence of trapped surfaces

A crucial quantity for the following discussion is the expansion $\theta_{(l)} = h^{ab}\nabla_a l_b$ of outgoing null rays for 2-surfaces \mathcal{S} in an interior vicinity of the horizon, where h is the interior metric of \mathcal{S} and l is the vector field describing outgoing null rays. In order to analyze $\theta_{(l)}$ inside the black hole, we introduce horizon-penetrating coordinates $(R, \theta, \tilde{\varphi}, \tilde{t})$, in which the metric is *regular* at the horizon \mathcal{H} :

$$d\tilde{t} = dt + \frac{T(R)}{\Delta} dR, \quad d\tilde{\varphi} = d\varphi + \frac{\Phi(R)}{\Delta} dR, \quad \Delta := 4(R^2 - r_h^2). \quad (9)$$

The free functions T and Φ are chosen in such a way that

$$a(R, \theta) := \frac{4\hat{\mu}\hat{u} - T^2}{\Delta}, \quad b(R, \theta) := \frac{\Phi - \omega T}{\Delta} \quad (10)$$

are regular at $R = r_h$. Furthermore, we require $a > 0$ in order to guarantee that $\tilde{t} = \text{constant}$ is a *spacelike* surface. As a consequence, T has to obey the conditions

$$T = 2\sqrt{\hat{\mu}\hat{u}} = \frac{4r_h}{\kappa} \quad \text{for } R = r_h, \quad T > 2\sqrt{\hat{\mu}\hat{u}} \quad \text{for } R < r_h. \quad (11)$$

We thus obtain the regular line element⁴

$$ds^2 = \left(\frac{a}{\hat{u}} + \hat{u}b^2 \sin^2 \theta\right) dR^2 - \frac{\Delta}{\hat{u}} d\tilde{t}^2 + \hat{u} \sin^2 \theta (d\tilde{\varphi} - \omega d\tilde{t})^2 + \hat{\mu} d\theta^2 \\ + 2\left(\frac{T}{\hat{u}} + \omega \hat{u}b \sin^2 \theta\right) dR d\tilde{t} - 2\hat{u}b \sin^2 \theta dR d\tilde{\varphi}. \quad (12)$$

In these coordinates, we calculate the expansion $\theta_{(l)}$ for a surface \mathcal{S} in a small interior neighborhood of the horizon, described by

$$\mathcal{S}: \quad R = r_h - \varepsilon \hat{r}(\theta), \quad \theta \in [0, \pi], \quad \tilde{\varphi} \in [0, 2\pi], \quad \tilde{t} = \text{constant}, \quad (13)$$

where $\varepsilon > 0$, $\hat{r} > 0$ and $\varepsilon \hat{r} \ll r_h$. We obtain

$$\theta_{(l)}(\theta) = \frac{\hat{T}}{2\sqrt{2\hat{\mu}\hat{u}} \sin \theta} \left[\frac{(\hat{r}, \theta \sin \theta)_{,\theta}}{\hat{r} r_h} - \frac{(\hat{\mu}\hat{u})_{,R}}{2\hat{\mu}\hat{u}} \sin \theta \right] \Big|_{\mathcal{H}} \varepsilon + \mathcal{O}(\varepsilon^2), \quad (14)$$

where we have used that T is of the form $T = 2\sqrt{\hat{\mu}\hat{u}}|_{\mathcal{H}} + \varepsilon \hat{T}$ with $\hat{T} = \hat{T}(\theta) > 0$ on \mathcal{S} ; see (11). (Note that $\theta_{(l)} = 0$ on the horizon ($\varepsilon = 0$).)

Following Booth and Fairhurst [3], we study the criterion $\delta_{\tilde{n}}\theta_{(\tilde{l})} < 0$ (in their notation) for the existence of trapped surfaces, i.e. we characterize a physically relevant, sub-extremal black hole through a negative variation of the expansion on the horizon in the direction of an ingoing null field \tilde{n} . In our formulation, this is equivalent to the existence of a surface \mathcal{S} with a negative expansion $\theta_{(l)}$. We first show the following lemma.

Lemma 3.1. *A necessary condition for the existence of trapped surfaces in the interior vicinity of the event horizon of an axisymmetric and stationary black hole is*

$$\int_0^\pi (\hat{\mu}\hat{u})_{,R} \Big|_{\mathcal{H}} \sin \theta d\theta > 0. \quad (15)$$

Proof. Let the surface \mathcal{S} , defined in (13) for sufficiently small ε , be trapped. Then $\theta_{(l)}$ is negative everywhere on \mathcal{S} , and so is the term in square brackets in (14) for all $\theta \in [0, \pi]$.

⁴ Note that for the Kerr solution with the mass M and the angular momentum J , we obtain Kerr-type coordinates (where the slices $\tilde{t} = \text{constant}$ are spacelike) by choosing $T = 4M(M + 2R)$ and $\Phi = \text{constant} = 2J/M$.

Thus, the integral of this term along the horizon \mathcal{H} is negative. Integrating by parts, and using that $\hat{\mu}\hat{u} = \text{constant} > 0$ on \mathcal{H} (see (7)), yields

$$\frac{1}{r_h} \int_0^\pi \frac{\hat{r}_{,\theta}^2}{\hat{r}^2} \sin \theta \, d\theta - \frac{1}{2\hat{\mu}\hat{u}} \int_0^\pi (\hat{\mu}\hat{u})_{,R} \sin \theta \, d\theta < 0. \quad (16)$$

Since the first integral is non-negative, we immediately obtain (15). \square

4. Calculation of p_J

Following the notation of [1], we express the angular momentum J and the horizon area A of the black hole by

$$J = \frac{1}{8\pi} \oint_{\mathcal{H}} m^{a;b} \, dS_{ab} = -\frac{1}{16} \int_0^\pi \hat{u}^2 \omega_{,R} \Big|_{\mathcal{H}} \sin^3 \theta \, d\theta = -\frac{r_h^2}{4} \int_{-1}^1 V e^{2U} (1-x^2) \, dx, \quad (17)$$

$$A = 2\pi \int_0^\pi \sqrt{\hat{\mu}\hat{u}} \Big|_{\mathcal{H}} \sin \theta \, d\theta = 4\pi \sqrt{\hat{\mu}\hat{u}} \Big|_{\mathcal{H}} = 4\pi r_h^2 e^{2U(1)}, \quad (18)$$

where $x := \cos \theta$, $U(x) := \left[\frac{1}{2} \ln(r_h^{-2}\hat{u}) \right] \Big|_{\mathcal{H}}$, $V(x) := \left[\frac{1}{4} \hat{u} \omega_{,R} \right] \Big|_{\mathcal{H}}$ and m^a is the Killing vector corresponding to axisymmetry. Note that for this formulation we have used conditions (7) and (8). As a consequence, p_J takes the form

$$p_J \equiv \frac{8\pi J}{A} = -\frac{1}{2} e^{-2U(1)} \int_{-1}^1 V e^{2U} (1-x^2) \, dx. \quad (19)$$

5. Reformulation in terms of a variational problem

In order to prove the inequality in question, we show that for $|p_J| \geq 1$ equation (15) is violated, i.e. that there are no trapped surfaces in the interior vicinity of the horizon.

Using (4) and (6), the integrand in (15) can be expressed as

$$(\hat{\mu}\hat{u})_{,R} \Big|_{\mathcal{H}} = \frac{2}{r_h} \hat{u}^2 \Big|_{\theta=0} \left[1 - \frac{1}{16} \omega_{,R}^2 \hat{u}^2 \sin^2 \theta - \frac{\hat{u}_{,\theta}}{2\hat{u}} \left(\frac{\hat{u}_{,\theta}}{2\hat{u}} + 2 \cot \theta \right) \right] \Big|_{\mathcal{H}}. \quad (20)$$

Hence, we can write (15) in terms of U , V and x as follows:

$$\frac{1}{2} \int_{-1}^1 [(V^2 + U'^2)(1-x^2) - 2xU'] \, dx < 1 \quad \text{with} \quad U' := \frac{dU}{dx}. \quad (21)$$

From (19), the violation of (15) for $|p_J| \geq 1$ can thus be formulated as the following implication, to be valid for any regular functions U , V defined on $[-1, 1]$ and satisfying the condition $U(-1) = U(1)$ [which follows from (8)]:

$$\left| \int_{-1}^1 V e^{2U} (1-x^2) \, dx \right| \geq 2 e^{2U(1)} \Rightarrow \frac{1}{2} \int_{-1}^1 [(V^2 + U'^2)(1-x^2) - 2xU'] \, dx \geq 1. \quad (22)$$

We now show that the validity of this implication holds provided that an appropriate functional I (defined below) cannot fall below 1.

Applying the Cauchy–Schwarz inequality to the first inequality in (22) we obtain

$$4 e^{4U(1)} \leq \left(\int_{-1}^1 V e^{2U} (1-x^2) \, dx \right)^2 \leq \int_{-1}^1 V^2 (1-x^2) \, dx \int_{-1}^1 e^{4U} (1-x^2) \, dx. \quad (23)$$

Given this inequality, we replace the term $\int V^2(1 - x^2) dx$ in the second inequality in (22) and see that

$$I[U] := \frac{1}{2} \int_{-1}^1 [U'^2(x)(1 - x^2) - 2xU'(x)] dx + \frac{2e^{4U(1)}}{\int_{-1}^1 e^{4U(x)}(1 - x^2) dx} \geq 1 \tag{24}$$

is as a sufficient condition for the validity of the implication, with the functional $I: W^{1,2}(-1, 1) \rightarrow \mathbb{R}$ defined on the Sobolev space $W^{1,2}(-1, 1)$. Thus we have shown the following.

Lemma 5.1. *The inequality $|p_J| < 1$ for any sub-extremal axisymmetric and stationary black hole with surrounding matter holds provided that the inequality*

$$I[U] \geq 1 \tag{25}$$

is true for all $U \in W^{1,2}(-1, 1)$ with $U(-1) = U(1)$.

This reformulation has led us to the following *variational problem*: calculate the minimum of the functional I and show that it is not below 1.

6. Complete solution of the variational problem

Consider the functional

$$I_\varepsilon[U] := \frac{1}{2} \int_{-1+\varepsilon}^{1-\varepsilon} [U'^2(x)(1 - x^2) - 2xU'(x)] dx + \frac{2e^{4U(1-\varepsilon)}}{\int_{-1+\varepsilon}^{1-\varepsilon} e^{4U(x)}(1 - x^2) dx} \tag{26}$$

on $W^{1,2}(-1 + \varepsilon, 1 - \varepsilon)$, where $0 < \varepsilon \ll 1$ is a fixed real number. We use techniques from the calculus of variations to show that there exists a minimizer U_ε for I_ε in a suitable class with sufficiently large value $I_\varepsilon[U_\varepsilon]$. Following this investigation, we take the limit $\varepsilon \rightarrow 0$ and see that the claim of lemma 5.1 follows.

We now show the following statements:

- (i) The functional I_ε is well defined on the Sobolev space $\overset{\varepsilon}{W}^{1,2} := W^{1,2}(-1 + \varepsilon, 1 - \varepsilon)$ of functions U defined almost everywhere on $(-1 + \varepsilon, 1 - \varepsilon)$. This is due to the well-known proposition below.

Proposition 6.1 (theorem 2.2 in Buttazzo–Giaquinta–Hildebrandt [4]). *On any bounded interval $J \subseteq \mathbb{R}$, $W^{1,2}(J) \hookrightarrow C^0(\bar{J})$ compactly. Moreover, the fundamental theorem of calculus holds in $\overset{\varepsilon}{W}^{1,2}$.*

Here, we use the adapted inner product $\int_{-1+\varepsilon}^{1-\varepsilon} UV d\mu + \int_{-1+\varepsilon}^{1-\varepsilon} U'V' d\mu$ for $U, V \in \overset{\varepsilon}{W}^{1,2}$, where $d\mu(x) := (1 - x^2) dx$, which is equivalent to the ordinary one and thus makes $\overset{\varepsilon}{W}^{1,2}$ a Hilbert space.

For ease of notation set $X^\varepsilon := \{U \in \overset{\varepsilon}{W}^{1,2} | U(1 - \varepsilon) = 0\}$. Note that the functional I_ε is invariant under addition of constants. We will use this in order to restrict our attention to the Hilbert subspace X^ε on which $(U, V) := \int_{-1+\varepsilon}^{1-\varepsilon} U'V' d\mu$ is an equivalent inner product inducing the norm $\|U\| := (\int_{-1+\varepsilon}^{1-\varepsilon} U'^2 d\mu)^{1/2}$.

- (ii) I_ε is bounded from below. Using $0 \leq (\frac{x}{\sqrt{1-x^2}} - U'(x)\sqrt{1-x^2})^2 = \frac{x^2}{1-x^2} - 2xU'(x) + U'^2(x)(1 - x^2)$ we conclude that $I_\varepsilon[U] \geq -\frac{1}{2} \int_{-1+\varepsilon}^{1-\varepsilon} \frac{x^2}{1-x^2} dx =: C(\varepsilon) > -\infty$.

- (iii) Applying the Cauchy–Schwarz inequality to $\int_{-1+\varepsilon}^{1-\varepsilon} xU'(x) dx$, we obtain that $I_\varepsilon[U] \geq \frac{1}{2}\|U\|^2 + 2C(\varepsilon)\|U\|$ for any $U \in X^\varepsilon$ with $C(\varepsilon)$ as in (ii). Hence, for every $P \in \mathbb{R}$ there exists a $Q_P \in \mathbb{R}$ such that $I_\varepsilon[U] \geq P$ whenever $\|U\| \geq Q_P$. This is equivalent to *coercivity of the functional I_ε with respect to the weak topology on X^ε* .
- (iv) The functional I_ε is *sequentially lower semi-continuous (lsc) with respect to the weak topology in X^ε* . Recall that lower semi-continuity is additive and that the first terms can be dealt with by standard theory (see, e.g., [6]). For the last term, we use proposition 6.1 to deduce that $U_k \rightarrow U$ in $C^0([-1 + \varepsilon, 1 - \varepsilon])$. Whence there exists a uniform bound $D > 0$ of $\{U_k\}$ so that by Lipschitz continuity of the exponential map on $[-4D, 4D]$ with Lipschitz constant L , we have

$$\left| \int_{-1+\varepsilon}^{1-\varepsilon} e^{4U_k} d\mu - \int_{-1+\varepsilon}^{1-\varepsilon} e^{4U} d\mu \right| \leq 4L \int_{-1+\varepsilon}^{1-\varepsilon} |U_k - U| d\mu \leq 8L \|U_k - U\|_{C^0([-1+\varepsilon, 1-\varepsilon])} \xrightarrow{k \rightarrow \infty} 0. \tag{27}$$

The last term thus being lsc, we have shown I_ε to be lsc.

We can now show the *existence of a global minimizer for I_ε in a suitable class*: as we have seen in (ii), I_ε is bounded from below on X^ε . Hence for $a \in \mathbb{R}$ and $c > 0$ we can choose a minimizing sequence $\{U_k\}$ in the class $\mathcal{K}_{a,c}^\varepsilon := \{U \in X^\varepsilon \mid \int_{-1+\varepsilon}^{1-\varepsilon} e^{4U} d\mu = c, U(-1 + \varepsilon) = a\}$, with values tending to the infimum $i_{a,c}^\varepsilon$ of I_ε in $\mathcal{K}_{a,c}^\varepsilon$. By coercivity, $\{U_k\}$ is bounded and we can extract a weakly converging subsequence with the limit $U_{a,c}^\varepsilon \in X^\varepsilon$ by Hilbert space techniques (theorem of Eberlein–Shmulyan [6]).

The class $\mathcal{K}_{a,c}^\varepsilon$ is weakly sequentially closed by proposition 6.1, which can be shown as in (iv). Whence by (iv), $U_{a,c}^\varepsilon \in \mathcal{K}_{a,c}^\varepsilon$ satisfies $I_\varepsilon[U_{a,c}^\varepsilon] = i_{a,c}^\varepsilon$.

Set $\overset{\varepsilon}{W}_{a,0}^{1,2} := \{U \in \overset{\varepsilon}{W}^{1,2} \mid U(-1 + \varepsilon) = a, U(1 - \varepsilon) = 0\} \subset X^\varepsilon$ for any $a \in \mathbb{R}$. By the theory of Lagrange multipliers, each minimizer of I_ε in the class $\mathcal{K}_{a,c}^\varepsilon$ is a critical point of the functional

$$J_{\varepsilon,c} : \overset{\varepsilon}{W}_{a,0}^{1,2} \rightarrow \mathbb{R} : U \mapsto \frac{1}{2} \int_{-1+\varepsilon}^{1-\varepsilon} [U^2(x)(1 - x^2) - 2xU'(x)] dx + \frac{\lambda}{2} \left(\int_{-1+\varepsilon}^{1-\varepsilon} e^{4U} d\mu - c \right) \tag{28}$$

for some $\lambda \in \mathbb{R}$, which is well defined and sufficiently smooth by proposition 6.1. In other words, there is $\lambda := \lambda_{a,c}^\varepsilon \in \mathbb{R}$ such that $U := U_{a,c}^\varepsilon \in \mathcal{K}_{a,c}^\varepsilon$ satisfies

$$\int_{-1+\varepsilon}^{1-\varepsilon} [U'(x)\varphi'(x)(1 - x^2) - x\varphi'(x)] dx + 2\lambda \int_{-1+\varepsilon}^{1-\varepsilon} e^{4U(x)}\varphi(x)(1 - x^2) dx = 0 \tag{29}$$

for all $\varphi \in \overset{\varepsilon}{W}_{0,0}^{1,2}$. This can be restated to say that $U \in \overset{\varepsilon}{W}^{1,2}$ is a weak solution of

$$\begin{aligned} -U''(x)(1 - x^2) + 2xU'(x) + 1 + 2\lambda e^{4U(x)}(1 - x^2) &= 0 & \forall x \in (-1 + \varepsilon, 1 - \varepsilon), \\ U(-1 + \varepsilon) = a, & & U(1 - \varepsilon) = 0, & & \int_{-1+\varepsilon}^{1-\varepsilon} e^{4U} d\mu = c. \end{aligned} \tag{30}$$

Any weak solution $U \in \overset{\varepsilon}{W}^{1,2}$ of (30) can be shown to be smooth and to satisfy equation (30) strongly: for all $\varphi \in \overset{\varepsilon}{W}_{0,0}^{1,2}$, we can rewrite (29) as

$$0 = \int_{-1+\varepsilon}^{1-\varepsilon} \left[U'(x)(1 - x^2) - x - 2\lambda \int_{-1+\varepsilon}^x e^{4U} d\mu \right] \varphi'(x) dx, \tag{31}$$

where we used integration by parts and proposition 6.1. By the fundamental lemma of the calculus of variations, there is a constant $b \in \mathbb{R}$ such that

$$U'(x)(1-x^2) - x - 2\lambda \int_{-1+\varepsilon}^x e^{4U} d\mu = b \tag{32}$$

holds almost everywhere on $(-1 + \varepsilon, 1 - \varepsilon)$ as the integrand of (31) in square brackets is an L^1 -function. Solving for U' , we deduce the smoothness of U by a bootstrap argument (similar to p 462 in [5]). Differentiating equation (32), we get strong validity of (30). In particular, $U_{a,c}^\varepsilon$ is a smooth classical solution of the Euler–Lagrange equation of $J_{\varepsilon,c}$.

Interestingly, there exists an integrating factor for equation (30) and it can be solved explicitly: for

$$F(x) := -(1-x^2)^2 U'^2(x) + 2x(1-x^2)U'(x) + \lambda e^{4U(x)}(1-x^2)^2 - x^2 \tag{33}$$

we have

$$F'(x) = 2[x - (1-x^2)U'(x)][U''(x)(1-x^2) - 2xU'(x) - 1 - 2\lambda e^{4U(x)}(1-x^2)]. \tag{34}$$

Thus, it suffices to solve the first-order equation $F = \text{constant}$, which can be done with the substitution $W(x) := (1-x^2)^2 e^{4U(x)}$.

The unique smooth solution turns out to be

$$W(x) = \varepsilon^2(2-\varepsilon)^2 \left(\frac{e^{2\alpha \operatorname{artanh}(1-\varepsilon)} - \beta e^{-2\alpha \operatorname{artanh}(1-\varepsilon)}}{e^{2\alpha \operatorname{artanh} x} - \beta e^{-2\alpha \operatorname{artanh} x}} \right)^2, \tag{35}$$

where the Lagrange multiplier λ was eliminated using the condition $U(1-\varepsilon) = 0$. The complex integration constants α and β are implicitly given in terms of a and c via $U(-1+\varepsilon) = a$ and $\int_{-1+\varepsilon}^{1-\varepsilon} e^{4U} d\mu = c$. The corresponding value of I_ε is

$$I_\varepsilon[U_{a,c}^\varepsilon] = \frac{\alpha\beta}{2} \frac{(2-\varepsilon)^{4\alpha} - \varepsilon^{4\alpha}}{(1+\beta^2)[\varepsilon(2-\varepsilon)]^{2\alpha} - \beta[(2-\varepsilon)^{4\alpha} + \varepsilon^{4\alpha}]} + 1 - \varepsilon + \frac{1-\alpha^2}{2} \ln \frac{\varepsilon}{2-\varepsilon} + \frac{8\alpha[\varepsilon(2-\varepsilon)]^{2(\alpha-1)}}{[(2-\varepsilon)^{2\alpha} - \varepsilon^{2\alpha}]^2} \cdot \frac{(1+\beta^2)[\varepsilon(2-\varepsilon)]^{2\alpha} - \beta[(2-\varepsilon)^{4\alpha} + \varepsilon^{4\alpha}]}{(2-\varepsilon)^{4\alpha} - \varepsilon^{4\alpha}}. \tag{36}$$

We now study the limit $\varepsilon \rightarrow 0$. For any $U \in W^{1,2}(-1, 1)$, set $a_\varepsilon := U(-1+\varepsilon) - U(1-\varepsilon)$, $c_\varepsilon := \int_{-1+\varepsilon}^{1-\varepsilon} e^{4U-4U(1-\varepsilon)} d\mu$, $c := \int_{-1}^1 e^{4U} d\mu$. Then $U|_{(-1+\varepsilon, 1-\varepsilon)} - U(1-\varepsilon) \in \mathcal{K}_{a_\varepsilon, c_\varepsilon}^\varepsilon$ and thus $I_\varepsilon[U] \geq i_{a_\varepsilon, c_\varepsilon}^\varepsilon$. As we have seen, there is a minimizer $U_{a_\varepsilon, c_\varepsilon}^\varepsilon$ of I_ε in $\mathcal{K}_{a_\varepsilon, c_\varepsilon}^\varepsilon$ and we obtain $I_\varepsilon[U] \geq I_\varepsilon[U_{a_\varepsilon, c_\varepsilon}^\varepsilon]$. As discussed above, this minimizer is smooth and solves the Euler–Lagrange equation of the functional $J_{\varepsilon, c_\varepsilon}$; the unique smooth solution of this equation is given by (35). For $\varepsilon \rightarrow 0$ we have $\alpha_\varepsilon \rightarrow 1$ (otherwise I_ε diverges) and $\beta_\varepsilon \rightarrow -1$ (a consequence of $a_\varepsilon \rightarrow 0$ by (25)). Using these relations, we obtain $I_\varepsilon[U_{a_\varepsilon, c_\varepsilon}^\varepsilon] \geq C_{a_\varepsilon}^\varepsilon$ with $C_{a_\varepsilon}^\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. By definition of I and I_ε we see that

$$|I(U) - I_\varepsilon(U)| \leq \frac{1}{2} \int_{1-\varepsilon \leq |x| < 1} |U'^2(x)(1-x^2) - 2xU'(x)| dx + 2 \left| \frac{e^{4U(1)}}{\int_{-1}^1 e^{4U} d\mu} - \frac{e^{4U(1-\varepsilon)}}{\int_{-1+\varepsilon}^{1-\varepsilon} e^{4U} d\mu} \right|, \tag{37}$$

where the first term tends to 0 and the denominators of the latter tend to each other as $\varepsilon \rightarrow 0$ (the integrands are L^1 -functions, theorem of bounded convergence). The numerators of the latter term converge to each other by proposition 6.1. Thus, $I[U] \geq 1$.⁵

⁵ Interestingly, equality in both inequalities of (22) (and hence $I(U) = 1$) is achieved only for the horizon functions U and V of a *degenerate* black hole (e.g. extreme Kerr), cf [1].

Applying lemma 5.1, we have shown $|p_J| < 1$ for *sub-extremal* black holes. Together with the results about *extremal* black holes [1]⁶ we arrive at the following.

Theorem 6.2. *Consider spacetimes with pure gravity (no electromagnetic fields) and vanishing cosmological constant. Then, for every axisymmetric and stationary sub-extremal black hole with arbitrary surrounding matter we have that $8\pi|J| < A$. The equality $8\pi|J| = A$ holds if the black hole is degenerate (extremal).*

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⁶ With the presented *unique* smooth solution (35) of the differential equation (30), the assumption of equatorial symmetry as well as that of the existence of a continuous sequence in the proof presented in [1] can be abandoned, cf equation (35) in [1].