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# A new numerical approach to the oscillation modes of relativistic stars

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**ABSTRACT**
The oscillation modes of a simple polytropic stellar model are studied. Using a new numerical approach (based on integration for complex coordinates) to the problem for the stellar exterior we have computed the eigenfrequencies of the highly damped w-modes. The results obtained agree well with recent ones of Leins, Nollert and Soffel [*Phys. Rev. D* **48** 3467 (1993)]. Specifically, we are able to explain why several modes in this regime of the complex frequency plane could not be identified within the WKB approach of Kokkotas and Schutz [*Mon. Not. R. astr. Soc.* **255** 119 (1992)]. Furthermore, we have established that the "kink" that was a prominent feature of the spectra of Kokkotas and Schutz, but did not appear in the results of Leins *et al.*, was a numerical artefact. Using our new numerical code we are also able to compute, for the first time, several of the slowly damped (p) modes for the considered stellar models. For very compact stars we find, somewhat surprisingly, that the damping of these modes does not decrease monotonically as one proceeds to higher oscillation frequencies. The existence of low-order modes that damp away much faster than anticipated may have implications for questions regarding stellar stability and the lifetime of gravitational-wave sources. The present results illustrate the accuracy and reliability of the complex-coordinate method and indicate that the method could prove to be of great use also in problems involving rotating stars. There is no apparent reason why the complex-coordinate approach should not extend to rotating stars, whereas it is accepted that all previous methods will fail to do so.

**Key words:** Stars : neutron stars - Gravitational waves

## 1 INTRODUCTION

In the concluding remarks of their pioneering paper on nonradial oscillations of neutron stars Thorne and Campollataro (1967) described it as "*just a modest introduction to a story which promises to be long, complicated and fascinating*". The story has undoubtedly proved to be intriguing, and many authors have contributed to our present understanding of the pulsations of neutron stars. Of special interest are the attempts to calculate the eigenfrequencies of the so-called quasinormal modes of a stellar system. These are solutions to the perturbation equations which agree with the necessary boundary conditions at the centre and at the surface of the star, and at the same time behave as purely outgoing waves at spatial infinity. In general relativity pulsations of the stellar fluid will normally be damped due to the emission of gravitational radiation. Hence, the characteristic frequencies of the system will have complex values. The real parts correspond to physical oscillation frequencies, while the imaginary parts describe the damping rate of each oscillation mode.

The first attempts to compute outgoing-wave modes for relativistic stars concerned modes that have an analogue in Newtonian theory. It was believed (and verified by the early calculations for nonrotating stars) that the Newtonian modes of oscillation are shifted only slightly because of the coupling to gravitational radiation: Each characteristic frequency adopts a very small imaginary part. In practice, it is not straightforward to determine these modes with great accuracy. This difficulty became obvious with the first numerical calculations (Thorne 1969). An approach based on a variational principle devised by Detweiler and Ipser (1973) also suffers from this problem (Detweiler 1975). The difficulty of determining small imaginary parts with some accuracy was not overcome until much later when Lindblom and Detweiler (1983) combined numerical integration of the equations for the interior with a numerically integrated solution for the exterior. Using this technique they computed the fundamental (f) mode frequency for many realistic neutron star equations of state. However, they later realized that the fourth-order system of equations they had used for the stellar interior could become singular. Consequently, they suggested that a different choice of dependent variables be made (Detweiler & Lindblom 1985). Also worth mentioning in this context is a paper by Balbinski *et al.* (1985) where results obtained by the Lindblom-Detweiler scheme were found to be in satisfactory agreement with the predictions of the standard quadrupole formula.

The last few years have seen several important contributions to this field. In an impressive series of papers, Chandrasekhar and Ferrari have reformulated the problem. In their final system of equations all fluid perturbations have been eliminated (Chandrasekhar & Ferrari 1991a). This has the advantage that one can view the problem of gravitational waves being scattered by a star in a way similar to that used for black holes (Chandrasekhar 1983). In the second paper of the series, Chandrasekhar and Ferrari show that odd-parity (axial) modes can be excited by the even-parity (polar) modes in the slow rotation limit (Chandrasekhar & Ferrari 1991b). The coupling is due to the Lense-Thirring effect. In the following papers, they demonstrate that their algorithm for computing eigenfrequencies can be trusted so long as the imaginary part is considerably smaller than the real part (Chandrasekhar *et al.* 1991), that the odd-parity (axial) modes can be of importance for extremely compact stars (Chandrasekhar & Ferrari 1991c) and that the idea of a complex angular momentum can be useful in astrophysical situations (Chandrasekhar & Ferrari 1992). A summary of their work has been written by Ferrari (1992). Another recent reformulation of the equations governing a perturbed stellar model was proposed by Ipser and Price (1991). They discuss the relation between the Regge-Wheeler gauge (and the resulting equations) and the diagonal gauge used by Chandrasekhar and Ferrari (Price & Ipser 1991). In fact, they reformulate the stellar perturbation equations in such a way that all fluid perturbations can be eliminated also in the Regge-Wheeler gauge. An interesting question is whether these new formulations of the pulsation problem will make computation of mode-frequencies any easier. We will not approach that issue here, but feel that an investigation of that kind is of great importance and should be encouraged.

Another chapter of the story begins with the study of a very simple toy model. Kokkotas and Schutz (1986) suggested that a plausible model for a stellar system would be a finite string – representing the fluid of the star – coupled by means of a spring to a semi-infinite string – the dynamical spacetime. They found that such systems could accommodate a new family of oscillation modes. These would be associated with the gravitational degrees of freedom, rather than the pulsations of the fluid, and would be strongly damped. Recently, a slightly more realistic toy model led Baumgarte and Schmidt (1993) to much the same conclusions. Yet another toy model, suggested by Kokkotas (1985), shows a lot of similarities with the axial (odd parity) modes discussed by Chandrasekhar and Ferrari (1991) and has recently led to the discovery of a branch of strongly damped axial modes (Kokkotas 1994).

A computation of highly damped modes for a realistic stellar model is not trivial, however. The main difficulty involves numerically separating the ingoing and outgoing-wave solutions at spatial infinity: The ingoing solution dies exponentially as $r \to \infty$ while the outgoing one grows. This problem is well-known from studies of quasinormal modes for black holes (see Andersson *et al.* (1993) for a detailed discussion). It seems reasonable to assume that methods that have proved useful in studies of the black-hole problem can be adapted to the stellar situation. A series expansion approach based on a four-term recurrence relation – similar to that developed by Leaver for black holes (Leaver 1985) – was used by Kojima (1988) to verify that the strongly damped modes (referred to as w-modes because of their connection to the gravitational waves) indeed do exist for realistic stellar models. Kokkotas and Schutz (1992) used a WKB approach (essentially a geometrical optics assumption of no reflection of waves in the exterior vacuum) in their calculations of w-mode spectra for several models. Their main results have recently been verified by Leins, Nollert and Soffel (1993). In their calculations, Leins *et al.* employed two different approaches for the exterior: Leaver's continued fraction approach (Leaver 1985) and a Wronskian technique that has proved extremely powerful for Schwarzschild black holes (Nollert & Schmidt 1992).

Although the results obtained by Leins *et al.* generally agree with those of Kokkotas and Schutz there are some differences. Leins *et al.* found new modes with considerably smaller oscillation frequencies and also higher damping than those that had been found by the WKB-technique. Their calculations also suggest that a "kink" that was apparent in the spectra of Kokkotas and Schutz does not exist. At first sight, one would be tempted to believe that the calculations of Leins *et al.* are the most reliable since the two approaches they used to deal with the exterior do not, in principle, involve any approximations. This, however, does not explain why the WKB approach becomes less reliable as the oscillation frequency increases. In fact, this is contrary to all expectations: The geometrical optics argument should be valid for high frequencies.

In this paper we attempt to settle these matters. We will combine a numerical phase-amplitude approach (similar to that used by Andersson (1992) for black holes) to the exterior problem with the Lindblom-Detweiler (Detweiler & Lindblom 1985) scheme for the inside. The key idea is to separate ingoing and outgoing solutions by numerically calculating their analytic continuations to a place in the complex-coordinate plane where they have comparabe amplitudes. This is a new approach that could prove to be of great importance, especially for problems involving rotating stars. In such problems the exterior spacetime is not known analytically and previous methods (such as the WKB method of Kokkotas and Schutz (1992) and the two methods employed by Leins *et al.* (1993)) will consequently fail. But it seems likely that a method based on complexifying the coordinates will work even if the spacetime itself can only be approximated. In fact, the method proposed here may well prove to be the only one that remains useful for the strongly damped modes of rotating stars. The present approach also has the advantage that, although based on numerical integration – and in that sense it provides an arbitrarily high numerical precision – it solves for quantities that are directly comparable to those used in the WKB scheme of Kokkotas and Schutz. Hence, in this first application of the new method, we hope to understand the reason for the discrepancies between the results of Kokkotas and Schutz (1992) and Leins *et* *al.* (1993) and thus contribute further to the understanding of the oscillation modes of nonrotating relativistic stars.

## 2 NONRADIAL PULSATIONS OF A RELATIVISTIC STAR

In Regge-Wheeler gauge the perturbed metric for a relativistic stellar model takes the form (Detweiler & Lindblom 1985)

$$\begin{aligned} ds^2 &= -e^\nu \left(1 + r^\ell H_0 e^{i\omega t} Y_{\ell m}\right) dt^2 - \\ & 2i\omega r^{\ell+1} H_1 e^{i\omega t} Y_{\ell m} dt dr + \\ & e^\lambda \left(1 - r^\ell H_0 e^{i\omega t} Y_{\ell m}\right) dr^2 + \\ & r^2 \left(1 - r^\ell K e^{i\omega t} Y_{\ell m}\right) \left(d\theta^2 + \sin^2\theta d\phi^2\right) \ , \end{aligned} \quad (1)$$

where $H_0$, $H_1$ and $K$ are functions of $r$ only, since we have assumed a harmonic dependence on time. We also have the standard definition of $M(r)$:

$$e^{-\lambda} = 1 - \frac{2M}{r} \ , \quad (2)$$

which plays the role of an effective mass inside radius $r$.

As is well-known, a stellar model in equilibrium is described by the Tolman-Oppenheimer-Volkov equations [see for example equations (A3)-(A5) of Lindblom and Detweiler (1983)]. These are a system of three coupled first order differential equations that determine the mass $M$, the metric quantity $\nu$ and the pressure $p$. Their solution requires that an equation of state $\rho = \rho(p)$ for the energy density is specified. In the present investigation we will assume a simple polytropic equation of state

$$p = \kappa\rho^2 \ , \quad (3)$$

where $\kappa = 100\text{km}^2$ and we are using units in which $c = G = 1$ throughout this paper. This choice of equation of state may not be very realistic, but it simplifies the calculations somewhat. It also enables us to compare our results directly to those of Balbinski *et al.* (1985), Kojima (1988), Kokkotas and Schutz (1992) and Leins *et al.* (1993).

Inside the star, the perturbed fluid is described by a Lagrangian displacement $\xi^a$, where

$$\xi^r = r^{\ell-1} e^{-\lambda/2} W e^{i\omega t} Y_{\ell m} \ , \quad (4)$$

$$\xi^\theta = -r^{\ell-2} V e^{i\omega t} \frac{\partial}{\partial\theta} Y_{\ell m} \ , \quad (5)$$

$$\xi^\phi = -\frac{r^{\ell-2}}{\sin^2\theta} V e^{i\omega t} \frac{\partial}{\partial\phi} Y_{\ell m} \ . \quad (6)$$

On general grounds one would think that perturbations of a spherical star have four degrees of freedom. Two of these are associated with the fluid and the remaining two correspond to the gravitational waves. This means that the five functions $H_0$, $H_1$, $K$, $W$ and $V$ should not be independent. As was shown by Detweiler and Lindblom (1985) a system of equations that is free from singularities can be obtained by defining

$$X = \omega^2(\rho + p) e^{-\nu/2} V - r^{-1} p' e^{(\nu-\lambda)/2} W + \frac{1}{2}(\rho + p) e^{\nu/2} H_0 \ , \quad (7)$$

where a prime denotes a derivative with respect to $r$. (A misprint in equation (6) of Detweiler and Lindblom (1985) has been corrected here.) Then it follows, as a consequence of Einstein's equations, that

$$\begin{aligned} (2M + nr + Q) H_0 &= 8\pi r^3 e^{-\nu/2} X - \\ & \left[(n+1)Q - \omega^2 r^3 e^{-(\nu+\lambda)}\right] H_1 + \\ & \left[nr - \omega^2 r^3 e^{-\nu} - \frac{e^\lambda}{r} Q(2M - r + Q)\right] K \ , \end{aligned} \quad (8)$$

where

$$Q = M + 4\pi r^3 p \ , \quad (9)$$

and

$$n = \frac{1}{2}(\ell + 2)(\ell - 1) \ . \quad (10)$$

The interior problem now reduces to a system of four first order differential equations for $H_1$, $K$, $W$ and $X$ [equations (8)-(11) in Detweiler & Lindblom (1985)]. Only two of the four linearly independent solutions to this system are well-behaved at the centre of the star (at $r = 0$). Furthermore, the perturbed pressure must vanish at the surface ($r = R$), which implies that $X(R) = 0$. These conditions specify a single acceptable solution for each frequency $\omega$. Physically, this solution describes the response of the star when gravitational waves of the given frequency are incident upon it.

In the exterior of the star the fluid perturbations vanish, and the two metric perturbations $H_1$ and $K$ can be combined in such a way [see for example Fackerell (1971)] that one obtains a single second-order differential equation known as the Zerilli equation. This equation can be written

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_Z(r)\right] Z = 0 \ , \quad (11)$$

with the effective potential $V_Z$ given by

$$V_Z(r) = 2\left(1 - \frac{2M}{r}\right) \times \frac{n^2(n+1)r^3 + 3n^2 M r^2 + 9nM^2 r + 9M^3}{r^3(nr + 3M)^2} \ . \quad (12)$$

The tortoise coordinate $r_*$ is defined by

$$\frac{d}{dr_*} = \left(1 - \frac{2M}{r}\right) \frac{d}{dr} \ . \quad (13)$$

Clearly, this equation allows two linearly independent solutions. Far away from the star one of these can be identified as an outgoing wave, whereas the other describes an ingoing wave. For a general frequency, the physically acceptable solution for the interior of the star leads to a mixture of out- and ingoing waves at spatial infinity. The quasinormal modes of the system are those special frequencies for which no waves come in from infinity. In a sense, these are the frequencies at which the star can be expected to radiate "spontaneously".

## 3 THE PHASE-AMPLITUDE APPROACH

In any numerical approach it is imperative that the quantities under consideration are smooth and nicely behaving. If that is not the case, an integration scheme may require ridiculously small steps in order to achieve the desired accuracy. One standard way to avoid this difficulty when one has to solve for (say) an oscillating function – such as the wave-like solutions in the stellar problem – is to appeal to general properties of the solutions and express them in terms of slowly varying functions. In the following we adopt this approach to deal with the exterior problem for stars.

Although the tortoise coordinate $r_*$ appears naturally [cf. (11)] in a formal description of the perturbation equations it may obscure a numerical solution of the problem, especially as we plan to use complex coordinates. In order to avoid any difficulties we introduce a new dependent variable $\Psi$ according to

$$Z = \left(1 - \frac{2M}{r}\right)^{-1/2} \Psi \ . \tag{14}$$

Then we get a new differential equation (Andersson *et al.* 1993)

$$\left[\frac{d^2}{dr^2} + U(r)\right] \Psi = 0, \tag{15}$$

where

$$U(r) = \left(1 - \frac{2M}{r}\right)^{-2} \left[\omega^2 - V_Z(r) + \frac{2M}{r^3} - \frac{3M^2}{r^4}\right] \ . \tag{16}$$

One of the important properties of an equation such as (15) is that the Wronskian of any two linearly independent solutions must be a constant. This means that the general solution can always be constructed as a combination of two basic solutions that has the form

$$\psi^{\pm} = q^{-1/2} \exp\left[\pm i \int q dr\right] \ , \tag{17}$$

where the function $q(r)$ is a solution to the nonlinear equation

$$\frac{1}{2q}\frac{d^2q}{dr^2} - \frac{3}{4q^2}\left(\frac{dq}{dr}\right)^2 + q^2 - U = 0 \ . \tag{18}$$

At first sight, it may seem as if we have replaced the relatively simple second-order differential equation (15) with a more complicated equation. In principle that may be true, but it should be quite obvious that the function $q$ – as determined from (18) – may be slowly varying even when the solution to (15) oscillates wildly.

Moreover, it is easy to make a connection to the WKB approximation here. Whenever $U$ is a slowly varying function of $r$ one can neglect the two derivatives in (18). That is, one can use $q \approx U^{1/2}$. Since that is the case for large $r$, we shall use this approximation to generate initial values for the integration of (18) far away from the star. This is done in such a way that $q \to \omega$ as $r \to +\infty$, *i.e.*, the function $\psi^+$ represents an ingoing wave whereas $\psi^-$ is outgoing.

## 4 STOKES PHENOMENON - WHY THE WKB APPROACH BREAKS DOWN

The purpose of this section is to study to what extent a WKB approximation of the Zerilli function at the surface of the star can be trusted. Specifically, we want to investigate whether a geometrical optics argument that the waves undergo no reflection in the exterior spacetime [as was assumed by Kokkotas and Schutz (1992)] can be used in this problem. If that is the case, one can simply express the exterior solution at the surface in terms of $U^{1/2}(R)$. In general, one would expect this kind of argument to be valid for high frequencies. In the case of really low frequencies, however, the situation becomes more obscure. Then waves can possibly be backscattered by the curvature of spacetime.

These qualitative expectations agree with the results of Leins *et al.* (1993) who found a few highly damped modes with a very small real part that Kokkotas and Schutz were unable to find. On the other hand, Leins *et al.* did not confirm the "kink" that occurred for relatively high frequencies in the WKB spectra. They suggested that this indicates that the WKB approach becomes less reliable as the oscillation frequency increases. This seems unreasonable: If the WKB argument is at all valid, the approximation should become more accurate as the oscillation frequency increases.

It is clear that the exact phase-amplitude approach outlined in the previous section provides us with the means to test the WKB assumption quantitatively. By comparing the numerically determined function $q$ to the approximation $U^{1/2}$ we can assess the reliability of the method used by Kokkotas and Schutz. (It should be noted that this approximation is not exactly that used by Kokkotas and Schutz (1992) where the Zerilli equation was approached directly. Nevertheless, a study such as the present one will provide results relevant also for that approximation.)

In approaching the problem for the stellar exterior numerically we must first devise a scheme that avoids a problem that arises for rapidly damped modes. The desired outgoing-wave solution to (15) will grow exponentially as $r \to +\infty$. This means that it is difficult to resolve the ingoing solution (that is exponentially small) given a finite numerical precision. This problem is well-known from studies of black-hole normal modes, and can be avoided by letting $r$ assume complex values [see Andersson (1992)]. Since $\psi^- \sim \exp(-i\omega r)$ as $r \to +\infty$ it is clear that the exponential divergence can be suppressed along a path in the complex $r$-plane. For large $\|r\|$ the preferred path is a straight line with slope given by $-\mathrm{Im}\ \omega/\mathrm{Re}\ \omega$. Such paths are parallell to the so-called anti-Stokes lines that play a crucial role in a complex WKB analysis (Andersson *et al.* 1993). The anti-Stokes lines are curves along which $\int U^{1/2} dr$ is a real quantity. In order to deal with the present problem we will integrate (18) along a straight line (with the prescribed slope) from a point in the asymptotic regime towards the stellar surface. One can show, either using an analytic continuation argument or a WKB analysis such as that of Araújo *et al.* (Araújo *et al.* 1993), that asymptotic conditions introduced for complex values of $r$ in the way described above are, in fact, identical to the desired outgoing-wave boundary condition on the real $r$-axis.

In a complex-coordinate WKB approach to (15) the so-called Stokes lines also play an important role [see the discussion by Andersson *et al.* (1993)]. These correspond to $\int U^{1/2} dr$ being purely imaginary. From each of the, possibly complex, zeros of $U^{1/2}$ – the transition points of the problem – emanate three such contours. When an approximate solution to (15) is continued across a Stokes line the character of the solution changes. This is called the Stokes phenomenon. Specifically it means that, if a certain linear combination of $\psi^+$ and $\psi^-$ (with $q$ replaced by $U^{1/2}$ ) represents the desired solution to (15) on one side of the Stokes line, another linear combination should probably be used on the opposite side.

**Figure 1.** Top: Comparing the numerically determined function $q$ (solid line) to the WKB approximation $U^{1/2}$ (dashed line). Only the real parts of the manifestly complex functions are shown, but the result is similar for the imaginary parts. This example is for model 4 (see the beginning of section 6.1 for particulars) and $\omega M = 0.35 + 0.84i$. The WKB approach of Kokkotas and Schutz cannot be trusted in this case. The numerical calculation towards the surface of the star is performed along a straight line at an angle of roughly -67 degrees from the real coordinate axis. This corresponds to the asymptotic direction of the so-called anti-Stokes lines. The Stokes phenomenon gives rise to oscillations in $q$ as one gets close to the surface. Bottom: For comparison we show results of a similar study on the real $r$-axis. It should be emphasized that the case shown here is one of the most difficult, and that the agreement between the exact and the approximate function is generally good when Re $\omega >$ Im $\omega$.

Moreover, it is straightforward to show [see the Appendix in Andersson (1993)] that this effect is important also for numerical integration. If the integration of (18) is continued across a Stokes line the function $q$ may become oscillatory.

We thus have a simple diagnostic of the "no-reflection" assumption used by Kokkotas and Schutz (1992): If a Stokes line (emanating from one of the complex zeros of $U^{1/2}$) crosses the real $r$-axis between $r = R$ and $r = +\infty$ we should have reflection. For the stellar models considered here we find that this does indeed happen for the new modes discovered by Leins *et al.* (1993). For small oscillation frequencies a Stokes line crosses the real $r$-axis in the relevant interval and the Stokes phenomenon should be accounted for in an approximate analysis. It is, of course, possible that this would give rise to a very small correction. To test whether this is the case, we have compared $q$ as numerically determined from (18) to $U^{1/2}$ for several frequencies. A typical example – for one of the new modes that were identified by Leins *et al.* (1993) – is displayed in Figure 1. From this Figure it follows that the WKB method used by Kokkotas and Schutz (1992) cannot be trusted for modes with a very small real part and a relatively large imaginary part.

In order to illustrate the difficulties involved in numerical integration along the real $r$-axis – and thus the advantage of using complex coordinates – we also show results of a real-$r$ calculation in Figure 1. This specific example corresponds to the worst possible situation. The numerically computed function $q$ has a singularity close to the real $r$-axis. That this can be expected if one fails to account for the Stokes phenomenon is clear from the equations given in the Appendix of Andersson (1993). Note also that the magnitude of $q$ drops dramatically immediately after passing close to this singularity. When the surface of the star is reached $|q|$ has decreased more than ten orders of magnitude.

Our study shows that the WKB technique should be reliable for modes with large real parts. This conclusion would mean that the "kink" in the spectrum of Kokkotas and Schutz cannot yet be dismissed. However, we can proceed one step further and use the numerically determined function $q$ in the necessary matching at the stellar surface. This should provide an accurate way of dealing with the exterior problem, and it will hopefully enable us to conclude whether the "kink" is real or not.

## 5 AN ACCURATE CONDITION FOR NORMAL MODES

In principle, it is a simple task to derive a normal-mode condition based on the phase-amplitude approach. We want to match the numerical solution to the inside problem to the solution for the exterior at the surface of the star. That is, we require that the Zerilli function and its derivative be continuous at the surface $R$. Then we can use (14) and

$$\frac{dZ}{dr_*} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{d\Psi}{dr} - \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1/2} \Psi \ , \qquad (19)$$

which follows immediately from (13).

Let us first assume that the physically acceptable solution for the interior of the star corresponds to a mixture of out- and ingoing waves at spatial infinity. Then we have

$$\Psi = A_{\rm in}\psi^+ + A_{\rm out}\psi^- \ , \qquad (20)$$

outside the star, and we get from (17)

$$\frac{d\Psi}{dr} = A_{\rm in}\psi^+ \left[ iq - \frac{1}{2q}\frac{dq}{dr} \right] - A_{\rm out}\psi^- \left[ iq + \frac{1}{2q}\frac{dq}{dr} \right] \ . \qquad (21)$$

The next step is to solve (18) for a particular value of $\omega$ from a point far away from the star in the way described in the previous section. This means that the above expressions for $\Psi$ and its derivative in terms of $\psi^\pm$ will remain valid. Moreover, if we take the lower limit of integration in (17) to be the surface $R$ of the star (where we do the matching) we get

$$\psi^\pm(R) = q^{-1/2}(R) \ , \qquad (22)$$

which means that we need only determine $q$ and its derivative.

It should be pointed out that the amplitudes $A_{\rm in}$ and $A_{\rm out}$ as defined by (20) differ slightly from those that follow from the asymptotic behaviour of the Zerilli function;

$$Z \sim B_{\rm in} e^{i\omega r_*} + B_{\rm out} e^{-i\omega r_*} \ . \qquad (23)$$

The difference is essentially a phase-factor and is of no importance for a search for mode-frequencies. With our definition the amplitudes of the functions that asymptotically represent out- and ingoing waves are equal at the surface of the star [see (22)].

Straightforward algebra leads to

$$A_{\rm in} = -\frac{i}{2\sqrt{q(R)}}\left(1 - \frac{2M}{R}\right)^{-1/2} \times$$
$$\left\{ Z_S \left[ \left(1 - \frac{2M}{R}\right)\left( iq + \frac{1}{2q}\frac{dq}{dr}\right)_{r=R} + \frac{M}{R^2} \right] + Z_S' \right\} \ , \qquad (24)$$

and

$$A_{\rm out} = \frac{i}{2\sqrt{q(R)}}\left(1 - \frac{2M}{R}\right)^{-1/2} \times$$

$$\left\{ Z_S \left[ \left(1 - \frac{2M}{R}\right) \left(-iq + \frac{1}{2q}\frac{dq}{dr}\right)_{r=R} + \frac{M}{R^2} \right] + Z_S' \right\} , \quad (25)$$

where $Z_S$ represents the interior solution at the surface of the star, and a prime denotes a derivative with respect to $r_*$. However, since the solution is only determined up to a constant factor by the physical conditions it makes more sense to study

$$\frac{A_{\rm out}}{A_{\rm in}} = \frac{Z_S \left[ \left(1 - \frac{2M}{R}\right) \left(iq - \frac{1}{2q}\frac{dq}{dr}\right) - \frac{M}{R^2} \right] - Z_S'}{Z_S \left[ \left(1 - \frac{2M}{R}\right) \left(iq + \frac{1}{2q}\frac{dq}{dr}\right) + \frac{M}{R^2} \right] + Z_S'} . \quad (26)$$

A quasinormal mode of the stellar system corresponds to a singularity of this ratio as a function of $\omega$.

It may be worthwhile to clarify the meaning of $dq/dr$ in the above equations, since we are assuming that $r$ is complex in the exterior. In principle, one could simply interpret (letting $z$ represent the complex $r$) $dq/dr$ as $dq/dz$ but this may not be very practical. As mentioned previously, we integrate for the exterior solution along a straight line in the complex $z$-plane. This line is parametrized by the real distance $\rho$ (from the surface of the star) with constant phase angle $\theta$. Then $dz/d\rho = e^{i\theta}$ and $dq/d\rho$ is the derivative of $q$ with respect to the real parameter. Thus we get

$$\frac{dq}{dr} = \frac{dq}{dz} = e^{-i\theta}\frac{dq}{d\rho} \quad (27)$$

which is easily implemented numerically.

## 6 DISCUSSION OF NUMERICAL RESULTS

The main purpose of the present study is to understand the discrepancies between the results of Kokkotas and Schutz (1992) and those of Leins *et al.* (1993). Specifically, we want to find the "new" modes that seem to exist for very small oscillation frequencies and large damping (Leins *et al.* 1993). We want to see if our ad hoc explanation for why those modes could not be found when a WKB approach to the exterior problem was used (see section 4) is correct, *i.e.,* that one must account for Stokes phenomenon for frequencies close to the imaginary $\omega$-axis. We also feel that it is imperative that we check the possible existence of the "kink" that Kokkotas and Schutz (1992) found in their spectra, especially since this feature was not found by Leins *et al.* (1993).

### 6.1 Highly damped (w) modes

We have performed detailed calculations for the four models studied by Kokkotas and Schutz (1992). A sample of the numerical results for model 4 (that has characteristics: $\rho_c = 10^{16}$ g/cm$^3$, $R = 6.465$ km and $M = 1.3M_\odot$, *i.e.,* $2M/R = 0.594$) can be found in Table 1.

**Table 1.** A sample of characteristic frequencies for stellar modes associated with model 4 of Kokkotas and Schutz (1992). The highly damped (w) modes are compared to the slowly damped (p) modes. The results for w-modes are in good agreement with those of Leins *et al.* (1993) The frequencies found for the f-mode and the first of the p-modes are in perfect agreement with those obtained by Kojima (1988). The "new" w-modes found by Leins *et al.* are indicated by † (the imaginary part of the one in parenthesis is too large for our program to provide truly reliable results, but there does seem to be a mode there). The same data is shown in Figure 2. All entries are given in units of $M^{-1}$. The given damping rates for the slowly damped modes are accurate to a few parts in $10^7$.

| Highly damped modes | | | Slowly damped modes | | |
|---|---|---|---|---|---|
| Re $\omega$ | Im $\omega$ | | Re $\omega$ | Im $\omega$ | |
| (0.142 | 1.286) | † | | | |
| | | | 0.171 | $6.21 \times 10^{-5}$ | $(f)$ |
| 0.353 | 0.838 | † | 0.344 | $2.2 \times 10^{-6}$ | $(p_1)$ |
| 0.471 | 0.056 | | 0.502 | $4.05 \times 10^{-5}$ | |
| 0.559 | 0.384 | † | | | |
| 0.654 | 0.164 | | 0.658 | $3.3 \times 10^{-6}$ | |
| 0.892 | 0.227 | | 0.810 | $5 \times 10^{-7}$ | |
| | | | 0.960 | $4 \times 10^{-7}$ | |
| 1.128 | 0.262 | | 1.100 | $5 \times 10^{-7}$ | |
| 1.363 | 0.287 | | | | |
| 1.599 | 0.307 | | | | |
| 1.836 | 0.324 | | | | |
| 2.073 | 0.339 | | | | |
| 2.310 | 0.353 | | | | |
| 2.549 | 0.365 | | | | |
| 2.788 | 0.375 | | | | |

We set out on this investigation to verify the existence of modes with large damping and small oscillation frequencies that had been found by Leins *et al.* (1993), and also to find whether there is a "kink" in the w-mode spectrum or not. The first of these questions was readily answered as soon as we had combined the numerical integration approach for the exterior vacuum (see sections 3–5) with the numerical code that Kokkotas and Schutz had used for the interior problem. The modes found by Leins *et al.* (1993) do, indeed, exist. In view of our understanding of the failure of the WKB approach for these frequencies this makes sense. As is clear from Figure 1, the WKB approach does break down when the real part of the frequency is small.

Leins *et al.* (1993) assumed that the "kink" in the spectra of Kokkotas and Schutz (1992) was evidence that the WKB method fails also as Re $\omega$ increases. We have found that this is not the case. Our first calculations gave results in good agreement with those of Kokkotas and Schutz for high frequencies, with a pronounced "kink' in all spectra, and it was clear that the WKB method is reliable for high oscillation frequencies. Leins *et al.* (1993) argue that their approach to the exterior problem did not involve approximations and should therefore be considered as more reliable than the WKB approximation. Still, we found that our new numerical scheme gave results that did not agree well with those of Leins *et al.*. At the same time, both methods have proved to be reliable and providing results of high accuracy in the case of black holes. Hence, it seemed likely that they should both be trusted, and that any discrepancies, *eg.*, the "kink", were due to differences in the approach to the interior problem. Unfortunately, this meant that we faced a rather difficult situation. The approach of Detweiler and Lindblom (1985) has been used in all studies of highly damped oscillations of relativistic stars. Hence, any differences would be in the numerical codes and rather difficult to find.

With this in mind we scrutinized our code for the interior problem, and found points where it could be improved. The amended code is much more reliable, and numerically accurate, than the original one used by Kokkotas and Schutz. It also runs considerably faster. It turns out that most of the improvements have a minor effect on the numerical results, however. (The results quoted in Table 1 were, obviously, computed with the final version.) After testing the code we have a much clearer idea of its restrictions. Most importantly, we have found that the "kink" is intimately related to the choice of point close to the centre of the star where power series expansions are used to initiate the numerical integration. When choosing a starting point one introduces an artificial length-scale in the problem, and this scale is reflected by the oscillation frequency at which the "kink" occurs. This can easily be illustrated if one considers the eigenfunctions: As the oscillation frequency increases the number of nodes in each eigenfunction increases. One will eventually reach a frequency where nodes should occur for smaller radii than the one where the numerical integration is initiated. The failure of the power-series solutions to account for this feature of the physical solutions leads to the "kink" in the spectrum. By initiating the integration sufficiently close to the centre of the star one can ensure that this "kink" does not occur in the frequency regime of interest. In the calculations discussed here we initiated the integration at a point roughly corresponding to 2.5% of $R$. It seems likely that Leins *et al.* (1993) initiated their integration at a point closer to the centre than we did originally, and had they continued their calculation to higher oscillation frequencies they would have found a "kink" similar to that of Kokkotas and Schutz (1992). The "kinks" are numerical artefacts.

We also find that it is extremely difficult to avoid numerical noise in the eigenfunctions. This is mainly due to the fact that the eigenfunction $X$ decreases rapidly towards the surface of the star (where it should vanish). When solving for the interior of the star one matches a solution from the centre (that satisfies the physical condition of regularity) to one from the surface of the star at some intermediate point. In order to get a reliable result it is imperative that this matching is not done at too large a radius. We find that the interior matching should be performed for radii smaller than something like $0.25R$. Furthermore, the calculation gets more difficult as Im $\omega$ increases. This is due to the fact that $\|X\|$ is then many magnitudes smaller than any of the other three eigenfunctions $K, H_1$ and $W$. Thus it is difficult to compute $X$ with acceptable accuracy. For this reason the calculation is more difficult for less relativistic stellar models: The damping of the w-modes generally increases as the model gets less relativistic. Interestingly, this explains why the results of Kokkotas and Schutz (1992) for model 1 were not as good as for the other three models. In view of this fact we decided to restrict our detailed study to models 2–4 of Kokkotas and Schutz (1992), and also not to attempt a search for modes with Im $\omega M > 1$. We feel that the interior problem must be reformulated if a reliable search for modes with very large damping is to be made. In this context a study based on the alternative formulations of Chandrasekhar and Ferrari (1991a), or Ipser and Price (1991), may provide interesting results. However, this does not mean that our results should not be taken seriously. We believe that the present study is the most accurate one to date within the range of validity that we have indicated.

**Figure 2.** The p- and w-mode spectra for model 4, cf. Table 1. Note the absence of a "kink" for high oscillation frequencies in the w-mode spectrum, and that the damping does not decrease monotonically as one proceeds up the p-mode spectrum.

### 6.2 Slowly damped (p) modes

An interesting effect of the improvements that we made to our code for the interior problem is that it is now considerably more accurate for modes with very slow damping. Hence, it is not at all difficult to compute the f-mode for the various models. Furthermore, we are able to iterate for several of the first p-modes. As far as we know noone has attempted to compute these modes before, and the results that we obtain are somewhat surprising. The damping of the p-modes does not necessarily, as has been generally presumed, decrease rapidly as one proceeds to higher oscillation frequencies. In fact, for the most compact of our models (model 4) we find an intriguing behaviour. There is a low-order mode that damps away much faster than expected (see Table 1). This more rapidly damped mode occurs only for the most compact of our stellar models.

This is an interesting result that could be of potential importance for many of our astrophysical expectations, such as the life-time of gravitational-wave sources. Furthermore, methods used by other authors to study f-modes could be used to test its correctness. For model 4 the ratio of real to imaginary part for the relevant mode is not considerably different from that for the f-mode. Therefore, it seems likely that the program used by Detweiler and Lindblom (1985), or that of Kojima (1988), could be used to test our results. In fact, Dr Kojima and Dr Lindblom have both very kindly performed this calculation. The results they obtain for the p-modes of model 4 are in good agreement with those listed in our Table 1.

It may be worthwhile to comment on the difficulty of finding p-modes here. It is clear that a numerical code must be very robust and accurate if iteration for these slowly damped modes is to be at all possible. The imaginary part is typically at least six orders of magnitude smaller than the real part, and if one requires single-digit precision in the imaginary part one must therefore achieve at least six digits in the real part. Our amended code should be reliable for imaginary parts larger than $10^{-7}$.

In cases when the imaginary part of the mode-frequency is so small that iteration is not possible, one can always attempt to infer the real part from a graphical approach. We know that the modes correspond to the zeros of the asymptotic amplitude $A_{\rm in}$ (or the singularities of the ratio $A_{\rm out}/A_{\rm in}$). If such a zero is situated close to the real $\omega$-axis it should, in principle, be easy to distinguish in a plot of $\log \|A_{\rm in}\|$. This idea was recently exploited by, for example, Ferrari and Germano (1994). That it works nicely is illustrated in Figure 3. When generating this figure we

**Figure 3.** Illustration of the graphic approach for finding p-modes. We show $\log_{10} \|A_{\rm in}\|$ as a function of real $\omega M$ for model 4. A slowly damped mode corresponds to a narrow singularity, and all the p-modes listed in Table 1 can easily be distinguished. Peaks in this figure indicate the existence of the w-modes. The real parts of the w and p-modes are shown as vertical lines at the top and bottom of the figure, respectively.

normalised the perturbed quantities in such a way that the radial displacement $W$ is equal to 1 at the surface of the star for all frequencies. The slowly damped p-modes give rise to very narrow singularities that are easy to distinguish although the data in the figure correspond to $\Delta\omega M = 10^{-3}$. It is not very time consuming to create a figure with this kind of resolution, and it is certainly worthwhile. In fact, we can not only pin down all the p-modes given in Table 1, but also guess at which oscillation frequencies there will be a w-mode. It seems as if a peak in $\log \|A_{\rm in}\|$ is an indication of a w-mode higher up in the complex $\omega$-plane. We did not expect to find this feature in this kind of graph and the exact origin of these peaks is something of a mystery. Interestingly the peaks do not show up in similar graphs generated by the numerical code of Lindblom (private communication). It thus seems that the difference between the two approaches, *i.e.* between (i) matching the interior solution to the exterior one at the surface of the star and (ii) using the value of the Zerilli function at the surface as initial data for integration towards infinity where the solution is matched to an asymptotic form, is crucial. We believe that the information contained in this sort of graph is important to our understanding of the stellar pulsations and are presently investigating this issue further. Although we do not yet have the a clearcut explanation of the phenomenon it is worthwhile to make one final point. It is easy to show that the peaks in Figure 3 originate from the interior solution and not the exterior one. Remembering that the Zerilli function is generated from the two spacetime perturbations $K$ and $H_1$, this agrees well with the notion that the spacetime perturbations dominate the fluid ones (which set the normalisation for the figure) at the w-mode frequencies.

### 6.3 Are there different families of highly damped modes?

Leins *et al.* (1993) argued that they had found a new family of highly damped modes. Given the present evidence we have to assess whether that is the case or not. We have shown why Kokkotas and Schutz (1992) could not find modes with high damping and small oscillation frequencies with the WKB scheme they used. But surely, that in itself does not imply that any modes they could not find belongs to a new family? Different families of modes must be distinguished for clear physical reasons. The difference between the two families (p- and g-modes) of slowly damped modes is well established, and the w-modes are certainly distinct. But given the knowledge available at the moment it is not at all clear that a similar split of the highly damped modes into different classes makes sense.

Leins *et al.* (1993) put forward two arguments for why their new modes belong to a different family. The first relates to the number of nodes of the eigenfunctions corresponding to the various modes. They find that the number of nodes increase systematically as one progresses up the w-mode sequence, but that the situation is not that clear for the new modes. In doing this they claim to be studying the "amplitude" of the eigenfunctions. What they actually study is the number of nodes in the real part of each eigenfunction. An argument solely based on the real part of a manifestly complex eigenfunction clearly does not make much sense. From, for example, (1) it is evident that both the real and the imaginary part will be relevant. Hence, the nodes in the imaginary part of the eigenfunction must also be studied. When that is done the conclusion regarding the higher oscillation frequencies remains the same, but for the new modes the situation is somewhat clearer. The imaginary part of each eigenfunction may have a node even if the real part does not, and vice versa. However, although it may provide a useful diagnostic, the number of nodes in the eigenfunctions is not a very good measure for distinguishing different families of modes. It is well known [see for example chapter 17 in Cox (1980)] that different p-modes may have an identical number of nodes in the eigenfunctions, and there is no reason why that could not be the case also for highly damped modes.

However, Leins *et al.* (1993) argue that their new modes are different from the ones of Kokkotas and Schutz (1992) because of other features of the eigenfunctions. For the new modes the eigenfunctions die out towards the centre of the star, whereas the amplitude stay roughly constant for all values of $r$ for the old modes. This difference is similar to that between g- and p-modes (Cox 1980). For the g-modes – which correspond to oscillation frequencies smaller than that of the f-mode – the eigenfunctions are known to die out towards the centre of the star, whereas this is not the case for the p-mode eigenfunctions. Could it be that the perturbations generally die out towards the centre of the star for very small oscillation frequencies? If so, a distinction of different w-mode families based on this feature would not be satisfactory.

At the present time, it seems premature to divide the w-modes of relativistic stars into different "families". If such a classification is to make sense it must be based on clear physical principles, and our understanding of the highly damped stellar oscillations is still far from satisfactory. The issue could probably be resolved by further detailed studies of different families of stellar models (Leins *et al.* 1993). A vital piece of information that is still missing from the puzzle regards the possible existence of modes with very large imaginary parts. As mentioned previously we have found that the present approach to the interior problem suffers from numerical difficulties when Im $\omega M > 1$ or so. This is not to be taken as an indication that modes with larger imaginary parts do not exist. On the contrary, such modes may well exist and a mode survey that covers the entire complex frequency plane could produce interesting results.

## 7 CONCLUDING REMARKS

In order to understand the discrepancies between results obtained by Kokkotas and Schutz (1992) and Leins, Nollert and Soffel (1993), we have computed highly damped modes for relativistic stars. The stellar models considered are simple polytropes used in several previous investigations. We have outlined a new numerical approach to the problem for the exterior of the star. This new scheme is based on numerical integration for complex coordinates, and is similar to one that has proved extremely reliable for black-hole problems (Andersson 1992).

With this new approach to the stellar exterior we are able to find modes that were not identified by Kokkotas and Schutz (1992). These modes, that are highly damped and situated close to the imaginary frequency-axis, agree perfectly with modes found by Leins *et al.* (1993). That the WKB method employed by Kokkotas and Schutz failed to distinguish these modes can be understood if the so-called Stokes phenomenon (familiar from WKB theory) is accounted for.

We have managed to explain the occurence of a "kink" in the spectra of Kokkotas and Schutz: It is a numerical artefact due to the choice of point close to the centre of the star where power-series expansions are used to initiate the numerical integration. This conclusion was drawn after we had scrutinized the way that we dealt with the stellar interior. We also found other points where our original code could be improved. Although none of those changes affected the numerical results significantly, the program is now more robust and runs considerably faster. That it allows us to do accurate calculations is illustrated by the fact that we could iterate for several of the very slowly damped p-modes. This led to a rather surprising result: For the most compact of our stellar models (corresponding to $2M/R = 0.594$) the damping of the p-modes does not decrease monotonically as one proceeds to higher oscillation frequencies. There is a low-order mode that damps away at least ten times faster than anticipated. This result may significantly affect our expectations regarding, for example, stellar stability and the lifetime of gravitational-wave sources. Hence, it is of some importance that it be studied in more detail.

The new complex-coordinate method that we have employed in the present study seems promising also for rotating stars, *i.e.*, when the exterior spacetime is only known approximately. Since all previously suggested methods will surely fail to handle that difficulty this new approach could turn out to be of considerable importance. The anomalous p-mode damping that we have discovered here would be especially interesting if it exists for rotating star f-modes, because these limit stability. Such issues are, however, beyond the scope of the present investigation and we will return to them in the future.

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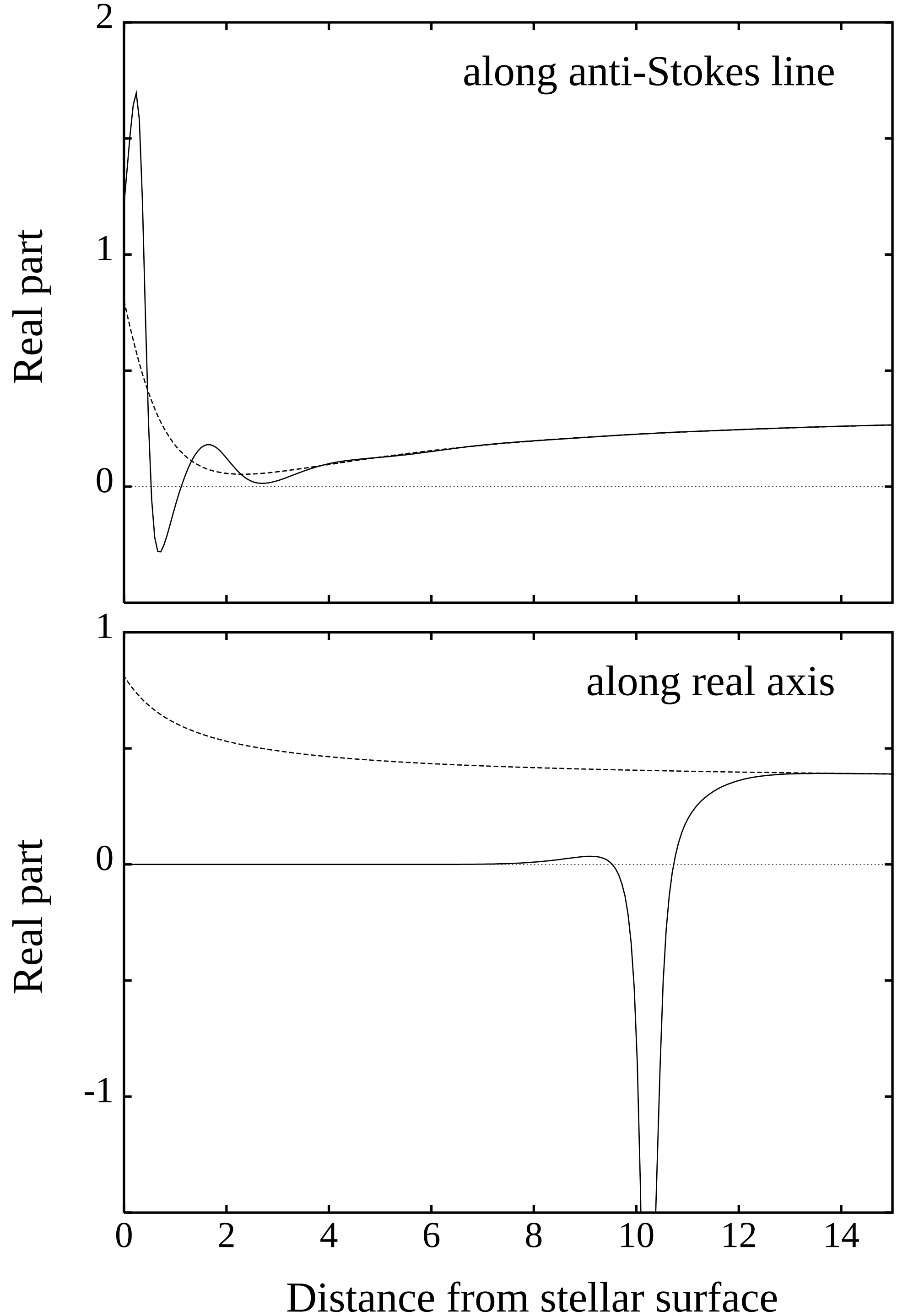
along anti-Stokes line
Real part
2
1
0
along real axis
Real part
1
0
-1
0
2
4
6
8
10
12
14
Distance from stellar surface

FIGURE 1

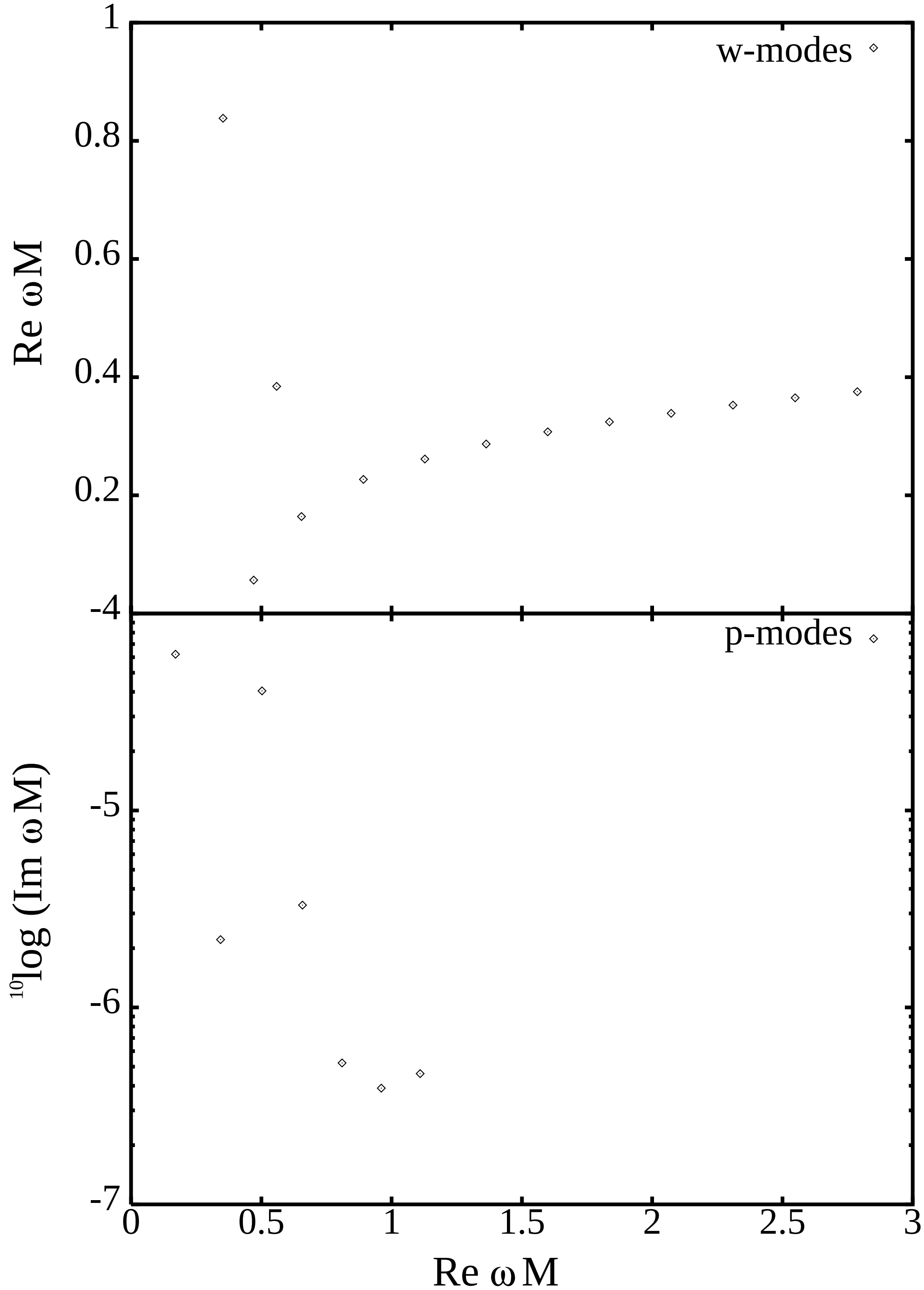
w-modes
p-modes
Re ωM
$^{10}$log (Im ωM)
Re ωM
1
0.8
0.6
0.4
0.2
-4
-5
-6
-7
0
0.5
1
1.5
2
2.5
3

FIGURE 2

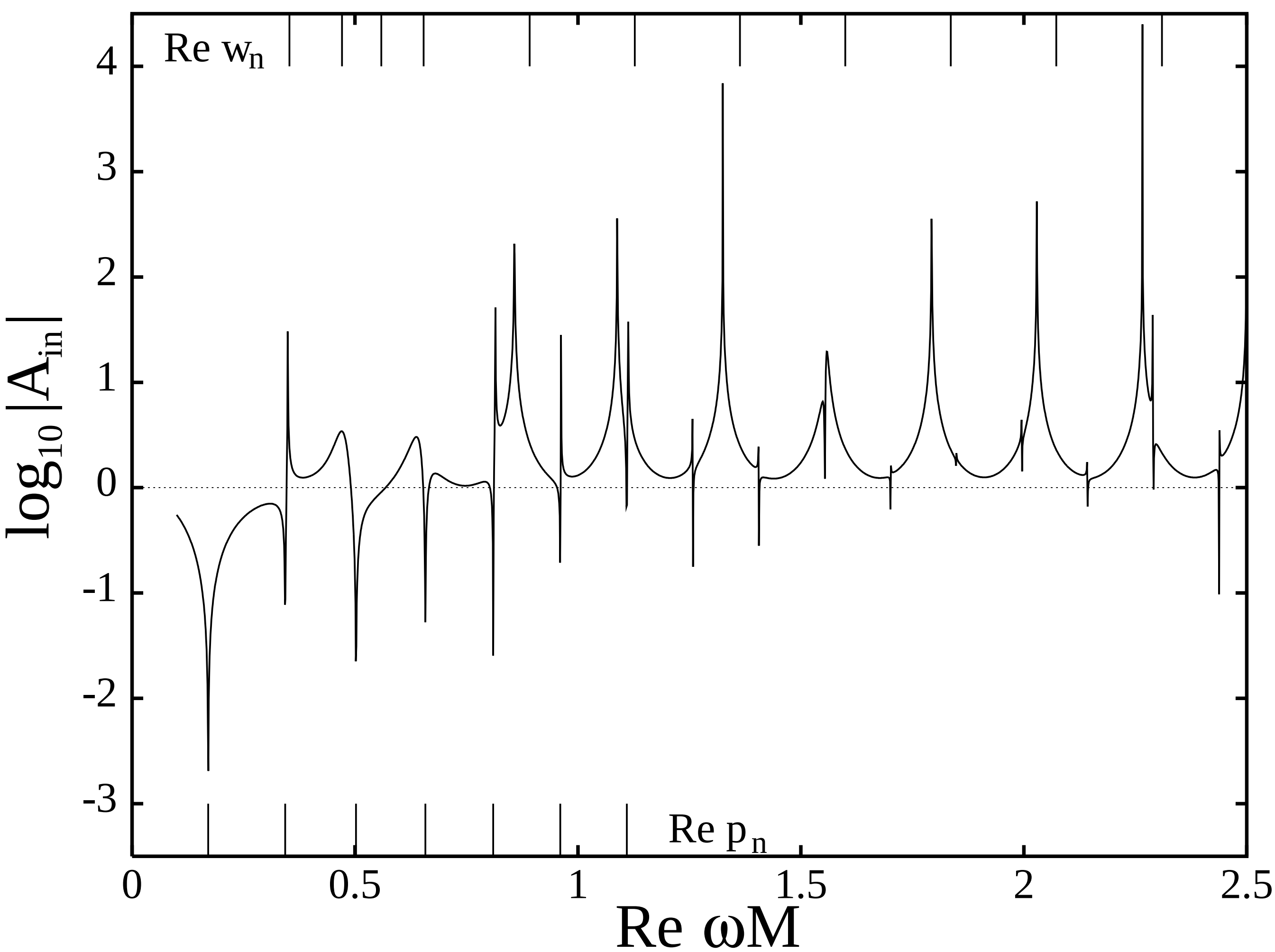


FIGURE 3