TIME-SYMMETRIC ADI AND CAUSAL RECONNECTION

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Abstract. We study the effect of a moving grid on the stability of the finite difference approximations to the wave equation. We introduce two techniques, which we call "causal reconnection" and "time-symmetric ADI" that together provide efficient, accurate and stable integration schemes for all grid velocities in any number of dimensions.

1 INTRODUCTION

In the numerical study of wave phenomena it is often necessary to use a reference frame that is moving with respect to the medium in which the waves propagate. In this paper, by studying the simple wave equation, we show that the consistent application to such a problem of two fundamental physical principles — causality and time-reversal-invariance — produces remarkably stable, efficient and accurate integration methods.

Our principal motivation for studying these techniques is the development of algorithms for the numerical simulation of moving, interacting black-holes. If we imagine a black hole moving "through" a finite difference grid then some requirements become clear. As the hole moves, grid points ahead of it will fall inside the horizon, while others will emerge on the other side. This requires grids that shift faster than light. Moreover, in situations when the dynamical time scale is large, one would like to be free of the Courant stability condition on time-steps, *i.e.* one wants to use implicit methods. Full implicit schemes require the inversion of huge sparse matrices. Alternating Direction Implicit (ADI) schemes reduce the computational burden by turning the integration into a succession of one-dimensional implicit integrations.

We will present here our most important results, leaving a more detailed derivation and analysis for a future paper (Alcubierre and Schutz (1992)).

2 THE WAVE EQUATION ON A MOVING GRID

We want to find a finite difference approximation to the wave equation using a grid that moves with an arbitrary non-uniform speed as seen in an inertial reference frame

(t, ξ^i). To represent this situation, we need to introduce a new coordinate system (t, x^i) that will be comoving with the grid

$$x^i = x^i(t, \xi^k). \tag{1}$$

In the new coordinates, the wave equation takes the form

$$(g^{ik} - \beta^i \beta^k) \frac{\partial^2 \phi}{\partial x^i \partial x^k} + \frac{2}{c} \beta^i \frac{\partial^2 \phi}{\partial x^i \partial t} - \Gamma^i \frac{\partial \phi}{\partial x^i} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{2}$$

where we have introduced the following quantities

- The spatial metric tensor g_{ij}.
- The shift vector β^i defined in the standard way (Misner, Thorne and Wheeler (1973))
$$x^i(\xi^j, t + dt) \approx x^i(\xi^j, t) - c \beta^i dt. \tag{3}$$
- The acceleration Γ^i defined in terms of the spatial Christoffel symbols
$$\Gamma^i := g^{kj} \Gamma^i_{kj} = -\frac{1}{\sqrt{g}} \left\{ \frac{\partial}{\partial t} (\sqrt{g} \beta^i) + \frac{\partial}{\partial x^j} [\sqrt{g} (g^{ij} - \beta^i \beta^j)] \right\}. \tag{4}$$

3 THE ONE DIMENSIONAL CASE

3.1 Finite Difference Approximation

In this case the metric, shift, and acceleration reduce to scalar functions

$$s^2(x,t) := g_{11}(x,t), \qquad \beta(x,t) := \beta^1(x,t), \qquad \Gamma(x,t) := \Gamma^1(x,t). \tag{5}$$

Using these expressions, (2) becomes

$$\left(\frac{1}{s^2} - \beta^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{2\beta}{c} \frac{\partial^2 \phi}{\partial x \partial t} - \Gamma \frac{\partial \phi}{\partial x} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{6}$$

For the finite difference approximation to this equation we employ the usual notation

$$\phi^j_i := \phi(i \Delta x, j \Delta t).$$

We define the first and second centered spatial differences as

$$\delta_x \phi^j_i := \phi^j_{i+1} - \phi^j_{i-1}, \qquad \delta^2_x \phi^j_i := \phi^j_{i+1} - 2\phi^j_i + \phi^j_{i-1}. \tag{8}$$

We can then write a second order accurate *implicit* finite difference approximation to (6) in the following way

$$\rho^2 \left(\frac{1}{s^2} - \beta^2\right) \left[\frac{\theta}{2} \left\{\delta_x^2 \phi_i^{j+1} + \delta_x^2 \phi_i^{j-1}\right\} + (1-\theta) \left\{\delta_x^2 \phi_i^j\right\}\right]$$

$$+\frac{\rho\beta}{2} \left[\delta_x \phi_i^{j+1} - \delta_x \phi_i^{j-1}\right] - \frac{\rho\,(c\,\Delta t)}{2} \Gamma \left[\delta_x \phi_i^j\right] - \left[\phi_i^{j+1} - 2\phi_i^j + \phi_i^{j-1}\right] = 0 \qquad (9)$$

with θ an arbitrary parameter that gives the weight of the implicit terms and ρ the "Courant" parameter given by

$$\rho := (c\,\Delta t)/\Delta x\,. \qquad (10)$$

The above finite difference approximation will be implicit whenever the shift vector is different from zero, even when $\theta = 0$. Therefore the use of implicit approximations for the spatial derivatives does not add any extra numerical difficulty.

3.2 Local stability

It is well known (Richtmyer and Morton (1967)) that the implicit approximation to the wave equation can be made unconditionally stable when $\beta = \Gamma = 0$ and $s = 1$ by taking $\theta \geq 1/2$. We have been interested in studying under what conditions this property is preserved with a shifting grid. The existence of a shift introduces a major difficulty: the coefficients in the equation generally depend on both position and time. This means that the stability analysis must be *local*: we will only consider the stability of the difference equation obtained from (9) by, at each point (x, t), taking the coefficients to be constant, with their values at that point.

When all the parameters are free to take any value, the resulting stability condition is very complicated, and it is then difficult to find its consequences analytically. We have therefore studied this condition numerically, in order to find regions of the parameter space in which the finite difference scheme is stable.

We have searched through many values of the parameters, and our local stability analysis suggests that the finite difference scheme will be stable for all time steps if

$$\theta \geq 1/2\,, \qquad |s\beta| < 1\,, \qquad \Gamma \text{ irrelevant.} \qquad (11)$$

3.3 Causal reconnection of the computational molecules

The causal structure of a grid shifting faster than the wave speed is particularly clear in the original (ξ, t) coordinates. In Figure 1 we see how, for a very large shift, the individual grid points move outside the light-cone. It then seems plausible that the instability found in the previous section should arise because the causal structure in not represented properly any more. This suggests that we should not build the computational molecules from grid points with fixed index labels, but instead from those points that have the closest *causal* relationship (Figure 2).

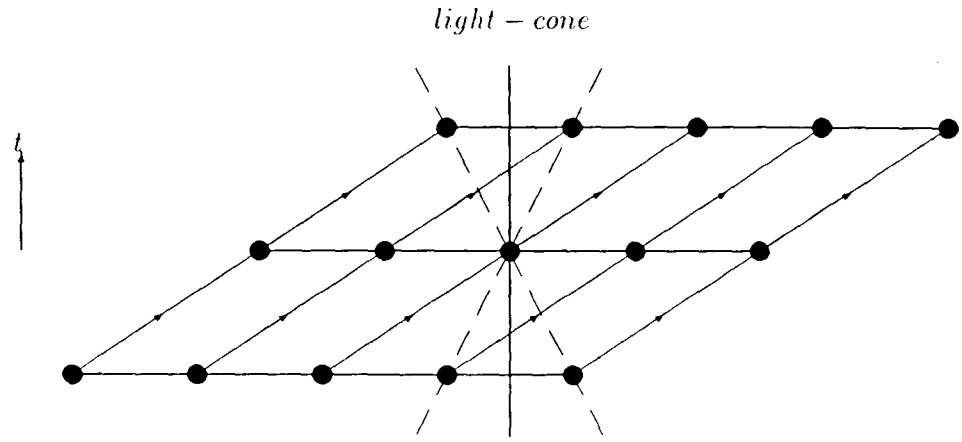

Figure 1: Grid moving faster than the waves.

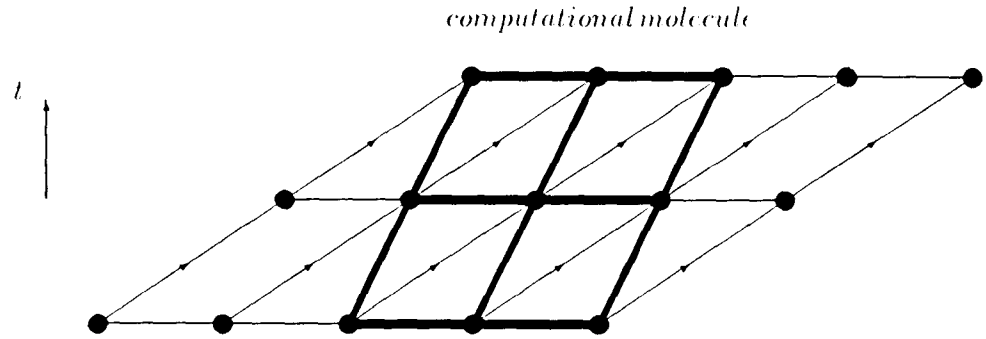

Figure 2: Causal computational molecule.

In Alcubierre and Schutz (1992) we introduce an algorithm to find the points that will form these *causal molecule*. Here we will assume that we have already found these points. The causal reconnection of the molecule is now implemented by introducing a new local coordinate system (x', t') adapted to the causal molecule. It can easily be seen that this new coordinate system (x', t') will move with respect to the old one (x, t) with a certain speed B at the intermediate time level, and with a constant acceleration A. In general the value of A and B will change from molecule to molecule, so the above change of coordinates must be repeated for each molecule.

In the primed coordinate system, the finite difference approximation will have the same form as before (equation (9)), except for the substitutions

$$\beta \longrightarrow \beta + \frac{B}{c}, \qquad \Gamma \longrightarrow \Gamma - \frac{A}{c^2}. \tag{12}$$

Causal reconnection can be implemented only when all the points in a given time level are inside the light-cone of some point in the previous time level. In the case when $\beta \neq \beta(x)$, this requirement takes the simple form

$$\Delta t \geq \frac{\max(s)\,\Delta x}{2c}. \tag{13}$$

This sets a *minimum* on Δt, and has a clear geometrical interpretation: the maximum proper distance between grid points ($\max(s)\,\Delta x$) must be smaller than the spread of the light-cone ($2c\,\Delta t$).

4 THE TWO-DIMENSIONAL CASE

4.1 Alternating Direction Implicit Methods

In matrix notation, the general form of the finite difference approximation to the wave equation in two dimensions is

$$\hat{Q}_2\,\phi^{j+1} = \hat{A}\,\phi^{j} + \hat{B}\,\phi^{j-1}, \tag{14}$$

with $\hat{Q}_2$, $\hat{A}$ and $\hat{B}$ spatial difference operators.

A straightforward generalization of the one-dimensional case will provide us with the most direct implicit approximation to the two-dimensional wave equation. We call this the "fully implicit" scheme. However, the numerical solution of this fully implicit scheme is considerably more time-consuming than in the one dimensional case. This is due to the fact that, if we have N grid points in each of n spatial directions, the matrix $\hat{Q}_n$ acting on ϕ^{j+1} will have N^n rows and columns. Most importantly, this matrix will *not* be tridiagonal. The matrix will still be sparse, but the number of operations involved in solving it may be very large indeed.

Alternating Direction Implicit (ADI) schemes reduce the numerical work involved in an n-dimensional problem by replacing the original large sparse matrix $\hat{Q}_n$ by one that can be factored into a product of tridiagonal matrices for each spatial direction. If we assume that we have the same number N of grid points in all directions, we will have to invert a series of N^{n-1} tridiagonal matrices of size $N \times N$ for each spatial dimension. This means that we will need only $O(nN^n)$ operations to solve the system. The reason that one can contemplate replacing the original operator $\hat{Q}_n$

with a different one is that the fully implicit scheme is only an approximation to the differential equation, so if we modify it by adding extra high-order terms that are of the same order as those neglected in the original approximation, the accuracy of the scheme will not be affected.

For our two-dimensional wave equation, the operator acting on ϕ^{j+1} turns out to be

$$\hat{Q}_2 := -1 + \frac{\rho}{2}\left(\beta^x \delta_x + \beta^y \delta_y\right) + \rho^2 \frac{\theta}{2}\left\{\left[g^{xx} - (\beta^x)^2\right]\delta_x^2 + \left[g^{yy} - (\beta^y)^2\right]\delta_y^2\right\}. \tag{15}$$

We want to add high-order terms to this expression to transform it into

$$\begin{aligned}\hat{Q}_2' &= \hat{Q}_x \hat{Q}_y \\ &:= -\left\{1 - \frac{\rho\beta^x}{2}\delta_x - \rho^2\frac{\theta}{2}\left[g^{xx} - (\beta^x)^2\right]\delta_x^2\right\} \\ &\quad \times \left\{1 - \frac{\rho\beta^y}{2}\delta_y - \rho^2\frac{\theta}{2}\left[g^{yy} - (\beta^y)^2\right]\delta_y^2\right\}.\end{aligned} \tag{16}$$

Let us define $\hat{S}$ to be the difference between these operators

$$\hat{S} := \hat{Q}_2' - \hat{Q}_2. \tag{17}$$

From this definition we find

$$\hat{S} = \hat{S}_{even} + \hat{S}_{odd}, \tag{18}$$

where we have separated terms linear and quadratic in β^i

$$\hat{S}_{even} = -\frac{\rho^2}{4}\beta^x\beta^y\delta_x\delta_y - \frac{\rho^4\theta^2}{4}\left(g^{xx} - (\beta^x)^2\right)\left(g^{yy} - (\beta^y)^2\right)\delta_x^2\delta_y^2,$$

$$\hat{S}_{odd} = -\frac{\rho^3\theta}{4}\left\{\beta^x\left(g^{yy} - (\beta^y)^2\right)\delta_x\delta_y^2 + \beta^y\left(g^{xx} - (\beta^x)^2\right)\delta_x^2\delta_y\right\}.$$

Now, we can't just add $\hat{S}(\phi^{j+1})$ to the finite difference approximation because in the limit when $\Delta x \to 0$ and $\Delta t \to 0$ we do not recover the original wave equation. There are many different ways to get around this problem

- Lees' first scheme. The most straightforward approach is that introduced by Lees in 1962 (Lees (1962), Fairweather and Mitchell (1965)) for the case of the ordinary wave equation on a fixed grid. In this method we add to the difference equation

$$\hat{S}\left(\phi^{j+1} - \phi^{j-1}\right). \tag{19}$$

  It is clear that as $\Delta t \to 0$ the extra term will vanish, and we will recover the original differential equation. To find the accuracy of the scheme, we

substitute the finite differences in the last expression for derivatives. We then see that the extra terms do not vanish as fast as the errors in the original approximation. Lees' first method is therefore only first-order accurate.

- Lees' second scheme. Another way to modify the equation is to add instead

$$\hat{S}\,(\phi^{j+1} - 2\phi^{j} + \phi^{j-1})\,. \tag{20}$$

Here again we recover the original differential equation in the limit $\Delta t \to 0$. It turns out that this method does not sacrifice accuracy: the introduced terms are of the same order as the original truncation error.

- Time-symmetric scheme. Both the original differential equation and the fully implicit scheme have the property of time-reversal invariance, that is, their form is preserved after the transformation

$$t \longrightarrow -t \qquad\qquad \beta^{i} \longrightarrow -\beta^{i}. \tag{21}$$

The operator $\hat{S}$ is not itself invariant: it contains linear and quadratic terms in β^{i}. Therefore, since Lees' first and second schemes both add terms in which $\hat{S}$ operates on an expression with a definite time-symmetry, neither scheme is time-reversal invariant. If we want to preserve the time-symmetry, we must allow the even and odd parts off $\hat{S}$ to act, respectively, on even and odd extra terms. That is, we add to the fully implicit scheme the term

$$\hat{S}_{e}\,(\phi^{j+1} - 2\phi^{j} + \phi^{j-1}) + \hat{S}_{o}\,(\phi^{j+1} - \phi^{j-1})\,, \tag{22}$$

Again, this scheme turns out to be just as accurate as the fully implicit method.

Whichever ADI method we choose, we will always produce an equation of the form

$$\hat{Q}'_{2}\,\phi^{j+1} = \hat{A}'\,\phi^{j} + \hat{B}'\,\phi^{j-1}\,, \tag{23}$$

with $\hat{A}'$ and $\hat{B}'$ new spatial finite difference operators whose specific form will depend on the method chosen. From the the definition of $\hat{Q}'_{2}$ we see that the last equation can be decomposed into a system of two coupled equations in the following way

$$\left\{1 - \frac{\rho\,\beta^{y}}{2}\,\delta_{y} - \rho^{2}\,\frac{\theta}{2}\left[g^{yy} - (\beta^{y})^{2}\right]\delta_{y}^{2}\right\}\phi^{j+1} := \phi^{*\,j+1}\,, \tag{24}$$

$$\left\{1 - \frac{\rho\,\beta^{x}}{2}\,\delta_{x} - \rho^{2}\,\frac{\theta}{2}\left[g^{xx} - (\beta^{x})^{2}\right]\delta_{x}^{2}\right\}\phi^{*\,j+1} = \hat{A}\,\phi^{j} + \hat{B}\,\phi^{j-1}, \tag{25}$$

where the first equation defines the so-called *intermediate value* $\phi^{*\,j+1}$.

4.2 Local Stability

The different ADI schemes differ not only on their accuracy, but also in their stability properties. We have studied numerically the local stability condition for each of these schemes. The details of this stability analysis, together with plots in parameter space of the stability properties of the different schemes, can be found in Alcubierre and Schutz (1992). Here we will only summarize our most important results:

- All three methods become unconditionally unstable as soon as one of the components of the shift is larger than the wave speed.
- For $\theta < 1/2$ all three methods become unstable for at least some value of the Courant parameter ρ, regardless of the value of the shift vector considered.
- For $\theta \geq 1/2$, Lees' first method turns unstable for at least some value of ρ whenever the grid speed is not aligned with one of the coordinate axis. The instabilities for speeds below that of the waves are mild, but nevertheless significant as our numerical experiments show.
- For Lees' second method the situation is even worst: the instabilities grow faster than for the first scheme.
- For $\theta \geq 1/2$, the time-symmetric ADI seems to be unconditionally stable for all values of the shift up to the wave speed. This scheme is therefore superior to both of Lees' methods when we have a moving grid.

5 NUMERICAL EXAMPLES

We have tested the difference methods introduced in the last sections in a number of different situations. In the one-dimensional case, we have tested causal reconnection using a grid that oscillates harmonically. We find that the direct approach, without causal reconnection, goes unstable as soon as the maximum grid speed is larger than the wave speed. Causal reconnection, on the other hand, has allowed us to use grid speeds of up to 15 times the wave speed without any instabilities.

The different ADI schemes have been tested in a uniformly moving two-dimensional grid. In accordance with our stability analysis, both Lees' methods turn unstable for quite small grid velocities. Time-symmetric ADI, however, remains stable as long as the grid speed doesn't reach the wave speed. In Figure 3 we see one such calculation, where we compare the results of the evolution of a gaussian wave packet on a moving grid for both Lees' first method and time-symmetric ADI. The grid is moving slower than the waves, but nevertheless Lees' first method shows a very clear instability after only 30 time steps. The calculation using time-symmetric ADI remains stable

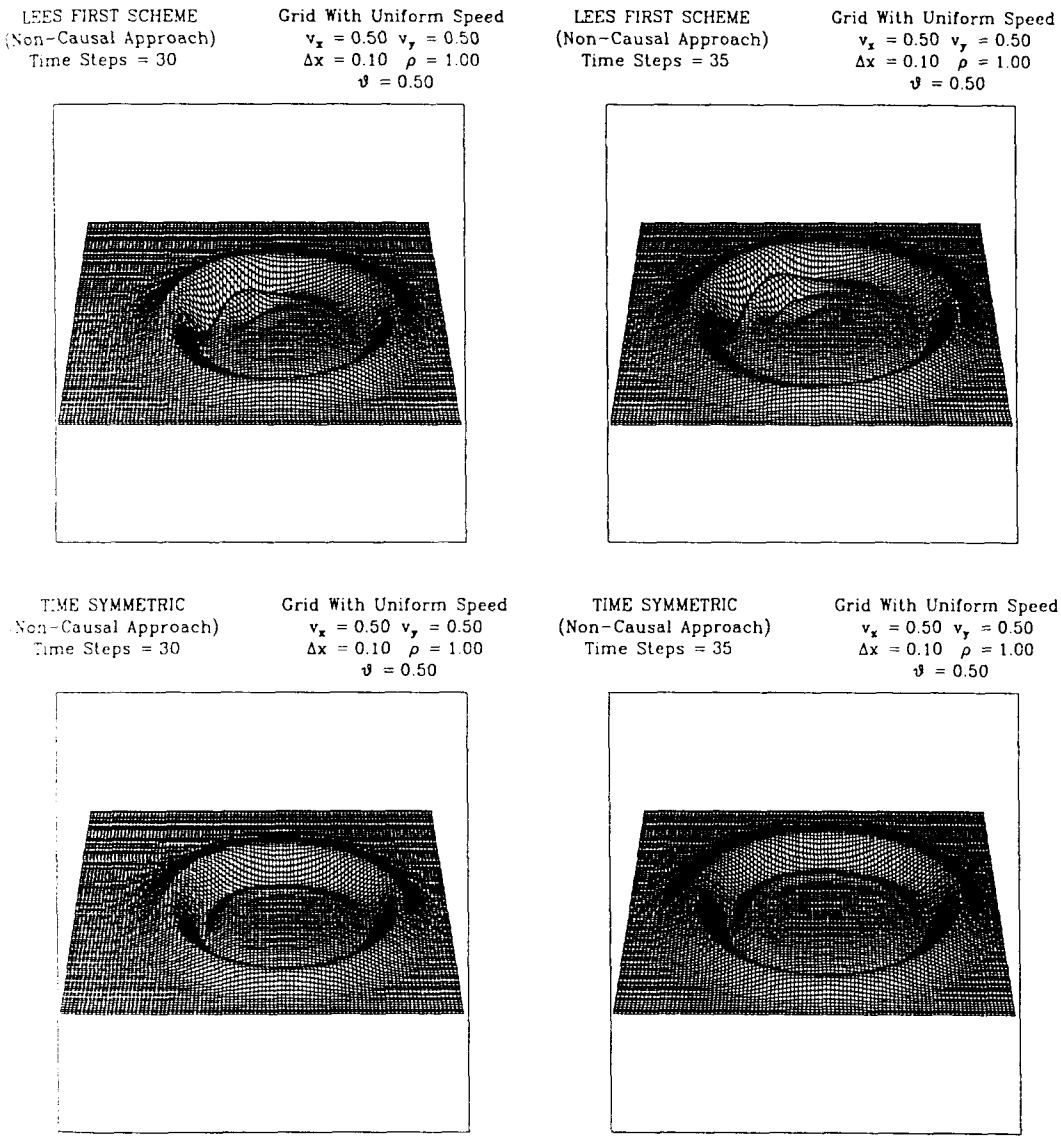

Figure 3. Uniformly moving grid: Lees' first method and time-symmetric ADI.

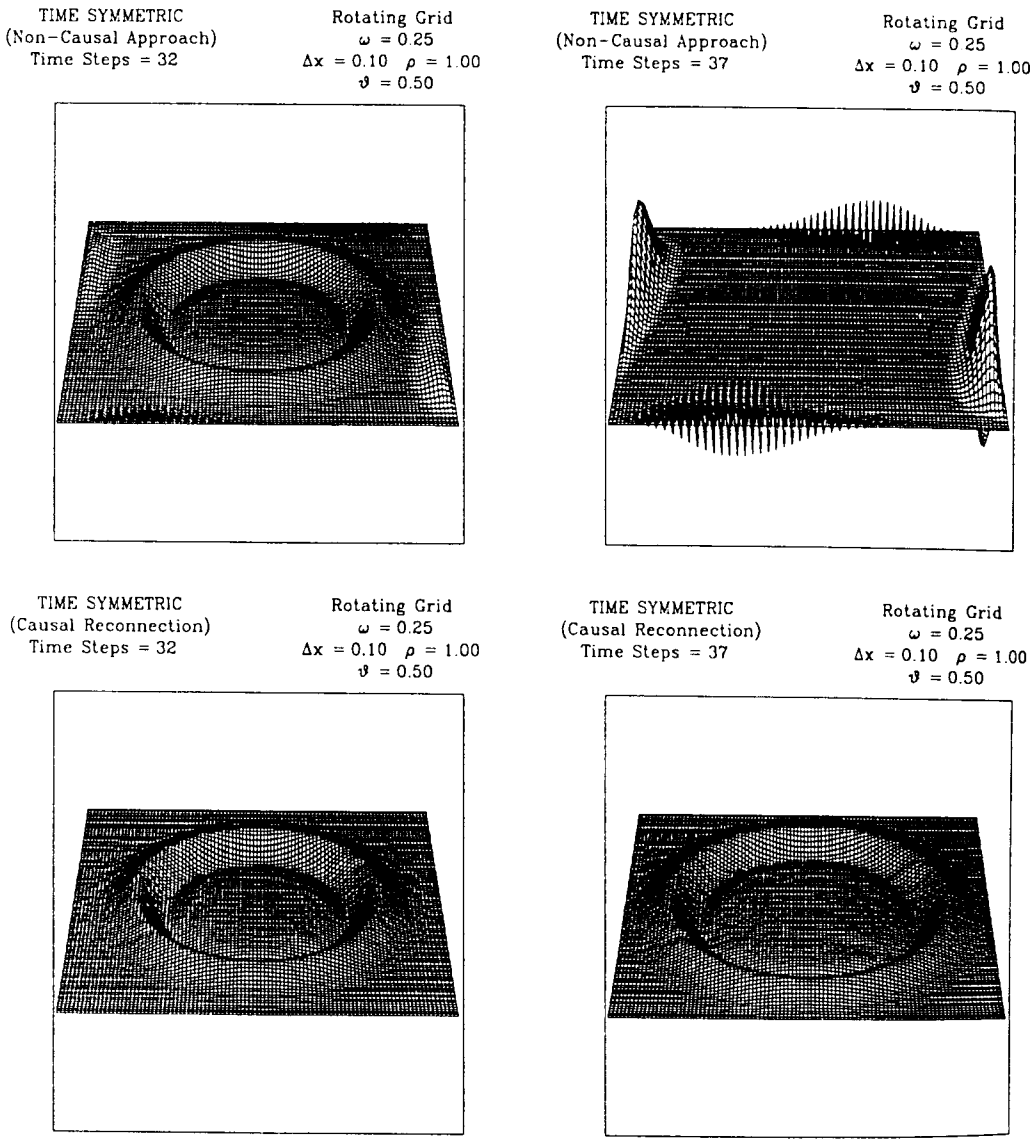

Figure 4. Rotating grid: non-causal method and causal reconnection.

Taking the time-symmetric ADI scheme as a starting point, we have tested causal reconnection in two dimensions using a rotating grid. We find that we can use grids whose edges move at many times the wave speed, without encountering any instabilities. In Figure 4 we show one example of a grid rotating in such a way that the speed at the edges is 1.25 of the wave speed, and we consider the evolution of a gaussian wave packet initially at rest at the center of the grid. When we don't reconnect the molecules, an instability forms after only 32 time steps, and grows so fast that 5 time steps later the original wave is no longer visible (the scale is set automatically to show the largest value). When we use causal reconnection, the instability is not present.

6 CONCLUSIONS

The wave equation is a prototype for more complex equations of mathematical physics. One would expect the instabilities we have found here to be *generic:* any numerical approximation to a hyperbolic system on a shifting grid should exhibit them. Only experience will show us just how well our cures for these generic instabilities transfer to more interesting equations. However, the instabilities we have described here are cured by the application of two clear physical principles, causality and time-reflection invariance. It seems clear that it would be asking for trouble *not* to incorporate these principles into the design of algorithms for the numerical integration of any fundamental physical equation. We are confident that causal reconnection and time-symmetric ADI will generalize easily to many problems in numerical relativity. These methods are stable, offer all the computational advantages of ADI schemes, and remain second order accurate.

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