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Asymptotics of solutions of the Einstein equations with positive cosmological constant.

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Abstract

A positive cosmological constant simplifies the asymptotics of forever expanding cosmological solutions of the Einstein equations. In this paper a general mathematical analysis on the level of formal power series is carried out for vacuum spacetimes of any dimension and perfect fluid spacetimes with linear equation of state in spacetime dimension four. For equations of state stiffer than radiation evidence for development of large gradients, analogous to spikes in Gowdy spacetimes, is found. It is shown that any vacuum solution satisfying minimal asymptotic conditions has a full asymptotic expansion given by the formal series. In four spacetime dimensions, and for spatially homogeneous spacetimes of any dimension, these minimal conditions can be derived for appropriate initial data. Using Fuchsian methods the existence of vacuum spacetimes with the given formal asymptotics depending on the maximal number of free functions is shown without symmetry assumptions.

1 Introduction

Spacetimes with accelerated expansion have come to play an important role in cosmology. The accelerating phase may be in the early universe (inflation) or at the present epoch (quintessence). The simplest way to produce a model with accelerated expansion which solves the Einstein equations is to introduce a positive cosmological constant. A good survey article on this topic is [19].

The fact that a positive cosmological constant leads to solutions of the Einstein equations with exponential expansion is associated with the term 'cosmic no hair theorem'. In the following we investigate possibilities of proving theorems related to these ideas. In the setting of formal series a satisfactory answer is obtained for the Einstein equations in vacuum or in the presence of a perfect fluid with linear equation of state. There are formal series solutions which have the expected asymptotic behaviour and which depend on the maximum number of free functions. This also holds for vacuum spacetimes in higher dimensions.

In the case of even space dimensions it is in general necessary to allow terms with logarithmic dependence on the expansion parameter. This throws some light on what is special about three space dimensions. These results are proved in Section 2.

While most of the results in three space dimensions obtained in Section 2 confirm the results of [18], one new phenomenon was observed. Evidence is obtained that for fluids with an equation of state stiffer than that of a radiation fluid inhomogeneous structures can be formed. This is reminiscent of the formation of spikes near the initial singularity in Gowdy spacetimes [17].

In Section 3 it is shown that in the vacuum case minimal assumptions on the asymptotics in the expanding phase imply that the spacetime has an asymptotic expansion of the form already exhibited as formal series. Unfortunately we do not know in general how to obtain these minimal assumptions starting from conditions on initial data. An exception to this is the case of three space dimensions where it is shown in Section 4 that the minimal assumptions can be deduced from results of Friedrich [4], [5] based on the conformal method. The minimal assumptions can also be verified in the case of certain spatially homogeneous vacuum spacetimes of any dimension, as shown in Section 5. In particular, there are genuine solutions of the Einstein equations whose asymptotics contain non-vanishing logarithmic terms. In Section 6 Fuchsian methods are applied to show the existence of vacuum spacetimes of any dimension with the asymptotics given in Section 2 and depending on the maximum number of free functions. Finally, Section 7 shows that in a model problem, the wave equation on de Sitter space, full information on asymptotics of solutions with arbitrary initial data can be obtained.

2 Perturbative solutions

A perturbative treatment of four-dimensional vacuum spacetimes with positive cosmological constant can be found in [18]. In that paper formal solutions are written down without any mathematical derivation being given. In this section a careful discussion of these formal power series solutions is presented. The analysis is generalized to vacuum spacetimes with positive cosmological constant in all dimensions. The expansion for perfect fluid spacetimes given in [18] is also revisited.

Consider the vacuum Einstein equations with cosmological constant Λ for a spacetime of dimension $n+1$ with $n \geq 2$. An $n+1$ decomposition with lapse equal to one and vanishing shift results in the constraint equations

$$R - k^{ab}k_{ab} + (\mathrm{tr}k)^2 = 2\Lambda \tag{1}$$

$$\nabla_a k^a{}_b - \nabla_b(\mathrm{tr}k) = 0 \tag{2}$$

and the evolution equation

$$\partial_t k^a{}_b = R^a{}_b + (\mathrm{tr}k)k^a{}_b - \frac{2\Lambda}{n-1}\delta^a{}_b \tag{3}$$

Here g_{ab} is the spatial metric with Ricci tensor R_{ab} and scalar curvature R, k_{ab} is the second fundamental form and indices are raised and lowered using g_{ab} and its inverse. Let $\sigma^a{}_b$ be the trace-free part of the second fundamental form and $\tilde{R}^a{}_b$ the trace-free part of the spatial Ricci tensor and define the following quantities:

$$\tilde{E}^a{}_b = \partial_t \sigma^a{}_b - [\tilde{R}^a{}_b + (\mathrm{tr} k)\sigma^a{}_b] \tag{4}$$

$$E = \partial_t(\mathrm{tr} k) - \left[R + (\mathrm{tr} k)^2 - \frac{2n\Lambda}{n-1}\right] \tag{5}$$

$$C = R - k^{ab}k_{ab} + (\mathrm{tr} k)^2 - 2\Lambda \tag{6}$$

$$C_a = \nabla_b k^b{}_a - \nabla_a(\mathrm{tr} k) \tag{7}$$

The Einstein equations are equivalent to the vanishing of the evolution quantities $\tilde{E}^a{}_b$ and E and the constraint quantities C and C_a. These quantities are linked by the following consistency conditions

$$\partial_t C = 2(\mathrm{tr} k)C - 2\nabla^a C_a - 2\sigma^a{}_b \tilde{E}^b{}_a + 2(1 - 1/n)(\mathrm{tr} k)E \tag{8}$$

$$\partial_t C_a = (\mathrm{tr} k)C_a - (1/2)\nabla_a C + \nabla_b \tilde{E}^b{}_a - (1 - 1/n)\nabla_a E \tag{9}$$

To do a perturbative analysis of equations (1)-(3) consider the formal power series

$$g_{ab} = e^{2Ht}(g^0_{ab} + g^1_{ab}e^{-Ht} + g^2_{ab}e^{-2Ht} + g^3_{ab}e^{-3Ht} + \ldots) \tag{10}$$

where H is a constant. The $n+1$ form of the Einstein equations are imposed on this expression in a suitable sense. It turns out that for consistency it is necessary to choose $H = \sqrt{2\Lambda/(n(n-1))}$ and so, in particular, Λ must be positive. Products of formal series are defined in the obvious way that the terms in the individual series are multiplied and the resulting terms with the same power of e^{-Ht} collected. The derivatives of a formal series with respect to the space and time variables are defined via term by term differentiation. In order to impose the Einstein equations it is also necessary to have a definition of the inverse g^{ab} of the formal power series metric g_{ab}. This can be done uniquely by requiring that the relation $g_{ab}g^{bc} = \delta^c_a$ holds. This allows the coefficient of order m in the series for g^{ab} to be expressed in terms of the coefficients in the series for g_{ab} up to order m. Setting $g^0_{ab} = \delta_{ab}$ and $g^m_{ab} = 0$ for $m > 0$ gives an exact solution of the Einstein equations. In the case $n = 3$ it is the de Sitter solution ([7], p. 125).

Given any tensor T, let $(T)_m$ denote the coefficient of e^{-mHt} in the expansion of T. With this notation $(g_{ab})_m = g^{m+2}_{ab}$. It follows from (10) that $(R^a{}_b)_m = 0$ for $m = 0$ and $m = 1$. It also follows directly from (10) that $(\mathrm{tr} k)_0 = -nH$ and $(\sigma^a{}_b)_0 = 0$. This is consistent with the vanishing of the coefficients of all evolution and constraint quantities for $m = 0$. The vanishing of $(E)_1$ and $(\tilde{E}^a{}_b)_1$ implies that $(\mathrm{tr} k)_1 = 0$ and $(\sigma^a{}_b)_1 = 0$. It follows that $(C)_1 = 0$ and $(C_a)_1 = 0$ and this ensures the consistency of the series up to order $m = 1$. Using the relation $\partial_t g_{ab} = -2g_{ac}k^c{}_b$ shows that $g^1_{ab} = 0$ and this in turn implies that $(R^a{}_b)_3 = 0$.

The relations between coefficients for $m \geq 2$ will now be written down. The summation indices p and q in the following formulae are assumed to be no less than two. The evolution equations (3) imply the recursion relations

$$(n-m)(\sigma^a{}_b)_m = H^{-1}\left[\sum_{p+q=m}(\sigma^a{}_b)_p(\mathrm{tr}k)_q + (\tilde{R}^a{}_b)_m\right] \tag{11}$$

and

$$(2n-m)(\mathrm{tr}k)_m = H^{-1}\left[\sum_{p+q=m}(\mathrm{tr}k)_p(\mathrm{tr}k)_q + (R)_m\right] \tag{12}$$

The Hamiltonian constraint (1) gives

$$(2n-2)(\mathrm{tr}k)_m = H^{-1}\left\{\sum_{p+q=m}[-(k^a{}_b)_p(k^b{}_a)_q + (\mathrm{tr}k)_p(\mathrm{tr}k)_q] + (R)_m\right\} \tag{13}$$

and the momentum constraint (2) gives

$$\nabla^0_a(k^a{}_b)_m - \nabla^0_b(\mathrm{tr}k)_m = \sum_{p+q=m}[(\Gamma^c_{ab})_p(k^a{}_c)_q - (\Gamma^a_{ac})_p(k^c{}_b)_q] \tag{14}$$

Here ∇^0 is the covariant derivative associated to g^0_{ab}. The consistency conditions (8) and (9) relating evolution and constraint quantities imply that if $(E)_k$, $(\tilde{E}^a{}_b)_k$, $(C)_k$ and $(C_a)_k$ all vanish for $k \leq m-1$ then

$$(2n-m)(C)_m = -2(n-1)(E)_m \tag{15}$$

$$(n-m)H(C_a)_m = -(1/2)\nabla_a(C)_m + \nabla^0_b(\tilde{E}^b{}_a)_m - (1-1/n)\nabla_a(E)_m \tag{16}$$

Consider first the case $n=3$. The form (10) of the series for g_{ab}, taking account of the vanishing of the coefficient g^1_{ab}, is contained in [18]. The following theorem formalizes some of the statements in [18]. Here smooth means C^∞.

Theorem 1 Let A_{ab} be a smooth three-dimensional Riemannian metric and B_{ab} a smooth symmetric tensor which satisfies $A^{ab}B_{ab}=0$ and $\nabla^a B_{ab}=0$, where the covariant derivative is that associated to A_{ab}. Then there exists a unique formal power series solution of the vacuum Einstein equations with cosmological constant $\Lambda > 0$ of the form (10) with $g^0_{ab}=A_{ab}$ and $g^3_{ab}=B_{ab}$. The coefficients g^m_{ab} are smooth.

Proof The coefficients $(k^a{}_b)_m$ determine the coefficients $(g_{ab})_m$ recursively. For substituting (10) into the relation $\partial_t g_{ab} = -2g_{ac}k^c{}_b$ gives

$$mHg^m_{ab} = 2g^0_{ac}(k^c{}_b)_m + 2\sum_{p+q=m} g^p_{ac}(k^c{}_b)_q \tag{17}$$

and hence an equation which expresses $(g_{ab})_{m-2}$ in terms of $(k^a{}_b)_m$ and lower order terms for any $m \geq 2$. Thus in order to prove the theorem it is enough to show that equations (11)-(14) determine the coefficients $(k^a{}_b)_m$ uniquely and that when the coefficients have been fixed in this way all the equations (11)-(14) are satisfied. The coefficient $(k^a{}_b)_m$ is determined by (11) and (12) for all $m \geq 2$ except $m = 3$ and $m = 6$. The coefficient $(k^a{}_b)_3$ is determined by using the condition that $g^3_{ab} = B_{ab}$. The coefficient $(\sigma^a{}_b)_6$ is determined by (11) while $(\mathrm{tr}k)_6$ is determined by (13). By construction the evolution equation (12) is satisfied for all values of m except possibly $m = 3$ and $m = 6$ while (11) is satisfied except possibly for $m = 3$. The fact that B_{ab} has zero trace ensures that (12) is satisfied while (11) is automatic for $m = 3$. It will now be shown by induction that (11)-(14) hold for all m. Except for $m = 6$ only (13) and (14) need to be verified. The equations (11)-(14) hold for $m = 1$. For $2 \leq m \leq 5$ the inductive step from $m - 1$ to m can be carried out as follows. When $m \neq 3$ the consistency condition (15) shows that $(C)_m = 0$ and then the consistency condition (16) shows that $(C_a)_m = 0$. That $(C)_3$ and $(C_a)_3$ are zero follows from the conditions on B_{ab} in the hypotheses of the theorem. Knowing that the equations (11)-(14) hold for $m \leq 5$ implies using (15) that $(E)_6 = 0$. By construction $(C)_6 = 0$ and then it is straightforward to obtain $(C_a)_6 = 0$ from (16). For $m \geq 7$ we can proceed as for $2 \leq m \leq 5$. This completes the proof.

Remark If P_{ab} and P denote the Ricci tensor and Ricci scalar of g^0_{ab} respectively, then $g^2_{ab} = H^{-2}(P_{ab} - (1/4)Pg^0_{ab})$, a relation given in [18].

The theorem just proved can be generalized directly to all larger odd values of n, as will now be shown.

Theorem 2 Let A_{ab} be a smooth n-dimensional Riemannian metric with n odd and B_{ab} a smooth symmetric tensor which satisfies $A^{ab}B_{ab} = 0$ and $\nabla^a B_{ab} = 0$, where the covariant derivative is that associated to A_{ab}. Then there exists a unique formal power series solution of the vacuum Einstein equations with cosmological constant $\Lambda > 0$ of the form (10) with $g^0_{ab} = A_{ab}$ and $g^n_{ab} = B_{ab}$. The coefficients g^m_{ab} are smooth.

Proof Let s be an integer such that $2s + 1 < n$ and $(k^a{}_b)_m = 0$ for all odd m with $m \leq 2s - 1$. If follows that $g^m_{ab} = 0$ for all odd m with $m \leq 2s - 1$ and that $(g^{ab})_m = 0$ for all odd m with $m \leq 2s+1$. Putting this information into the Ricci tensor shows that $(R^a{}_b)_m = 0$ for all odd m with $m \leq 2s+1$. Then (11) and (12) imply that $(k^a{}_b)_{2s+1} = 0$. It can then be proved by induction that $(k^a{}_b)_m = 0$ vanishes for all odd m with $m < n$. From this point on the proof is very similar to that of the previous theorem. The coefficients $(k^a{}_b)_m$ are uniquely determined for all values of m except $m = n$ and $m = 2n$. The coefficient $(k^a{}_b)_n$ is determined by using the condition that $g^n_{ab} = B_{ab}$. The coefficient $(\sigma^a{}_b)_{2n}$ is determined by (11) while $(\mathrm{tr}k)_{2n}$ is determined by (13). By construction the evolution equation (12) is satisfied for all values of m except possibly $m = n$ and $m = 2n$ while (11) is satisfied except possibly for $m = n$. The fact B_{ab} has zero trace ensures that (12) is satisfied while (11) is automatic for $m = n$. These statements make use of the fact that the odd order coefficients of $k^a{}_b$ of order less than n vanish. It will now be shown by induction that (11)-(14) hold for all m. Except for $m = 2n$ only (13) and (14) need to be verified. The equations (11)-(14) hold for $m = 1$. For $2 \leq m \leq 2n - 1$ the inductive step from $m - 1$

to m can be carried out as follows. When $m \neq n$ the consistency condition (15) shows that $(C)_m = 0$ and then the consistency condition (16) shows that $(C_a)_m = 0$. That $(C)_n$ and $(C_a)_n$ are zero follows from the conditions on B_{ab} in the hypotheses of the theorem. Knowing that the equations (11)-(14) hold for $m \leq 2n-1$ implies using (15) that $(E)_{2n} = 0$. By construction $(C)_{2n} = 0$ and then it is straightforward to obtain $(C_a)_{2n} = 0$. For $m \geq 2n+1$ we can proceed as for $2 \leq m \leq 2n-1$. This completes the proof.

The case where n is even is more complicated. The form (10) of the metric must be generalized to

$$g_{ab} = e^{2Ht}(g^0_{ab} + \sum_{m=0}^{\infty}\sum_{l=0}^{L_m}(g_{ab})_{m,l}t^l e^{-mHt}) \tag{18}$$

where L_m is a non-negative integer for each m and $L_m = 0$ for $m < n$. Given any tensor T with an expansion of the above type let $(T)_{m,l}$ denote the coefficient of $t^l e^{-mHt}$. As before when manipulating series they are differentiated term by term. The recursion relations for the expansion coefficients coming from the evolution equations generalize as follows, where the terms not written out explicitly are lower order in the sense that they can be expressed in terms of the coefficients of $k^a{}_b$ with m smaller:

$$(n-m)(\sigma^a{}_b)_{m,l} + H^{-1}(l+1)(\sigma^a{}_b)_{m,l+1} = \ldots \tag{19}$$

$$(2n-m)(\mathrm{tr}k)_{m,l} + H^{-1}(l+1)(\mathrm{tr}k)_{m,l+1} = \ldots \tag{20}$$

The terms on the right hand side not written out are obtained from the terms on the right hand side of equations (11) and (12) if the indices m, p and q are replaced by the pairs (m,l), (p,l_1) and (q,l_2), summing over $l_1 + l_2 = l$. The recursion relations implied by the constraints are identical except for the addition of an extra index l. In a similar way, the consistency conditions lead to

$$\begin{aligned}(2n-m)(C)_{m,l} + H^{-1}(l+1)(C)_{m,l+1} &= -2(n-1)(E)_{m,l} &(21)\\ (n-m)H(C_a)_{m,l} + H^{-1}(l+1)(C_a)_{m,l+1} &= -(1/2)\nabla_a(C)_{m,l}\\ +\nabla^0_b(\tilde{E}^b{}_a)_{m,l} &- (1-1/n)\nabla_a(E)_{m,l} &(22)\end{aligned}$$

assuming that $(E)_{k,l}$. $(\tilde{E}^a{}_b)_{k,l}$, $(C)_{k,l}$ and $(C_a)_{k,l}$ vanish whenever $k \leq m-1$.

By using the above relations it is possible to express $g^{ab}_0(g_{ab})_{n-2,0}$ as a function of g^0_{ab} and its spatial derivatives. We denote this schematically by $g^{ab}_0(g_{ab})_{n-2,0} = Z(g^0)$. Similarly, it is possible to write ${}^0\nabla^a(g_{ab})_{n-2,0} = \tilde{Z}_b(g^0)$.

Theorem 3 Let A_{ab} be a smooth n-dimensional Riemannian metric and B_{ab} a smooth symmetric tensor which satisfies $A^{ab}B_{ab} = Z(A)$ and $\nabla^a B_{ab} = \tilde{Z}_b(A)$, where the covariant derivative is that associated to A_{ab}. Then there exists a unique formal series solution of the vacuum Einstein equations with cosmological constant $\Lambda > 0$ of the form (18) with $g^0_{ab} = A_{ab}$ and $(g_{ab})_{n-2,0} = B_{ab}$. The coefficients $(g_{ab})_{m,l}$ are smooth.

Proof As a general principle, when determining coefficients for fixed m we start from $l = L_m$ and proceed to successively lower values of l. If n is odd then the existence follows from Theorem 2. Uniqueness for n odd in the wider class being considered in this theorem in comparison with Theorem 2 is obtained by a straightforward extension of the argument given in the proof of the latter. Consider now the case where n is even. By analogy with the proof of Theorem 2 it can be shown that $(k^a{}_b)_{m,l} = 0$ for m odd. For $m < n$ the coefficients are uniquely determined. The assumption that $L_m = 0$ in this range is thus unavoidable.The coefficients $(\mathrm{tr}k)_{n,l}$ are uniquely determined by (20) and vanish for $l > 0$. The coefficients $(\sigma^a{}_b)_{n,l}$ are uniquely determined for $l \geq 1$ and vanish for $l > 1$. The choice of B_{ab} determines $(\sigma^a{}_b)_{n,0}$. For $n < m < 2n$ equations (19) and (20) determine $(k^a{}_b)_{m,l}$. Equation (19) is used to determine $(\sigma^a{}_b)_{2n,l}$ while the analogue of (13) is used to determine $(\mathrm{tr}k)_{2n,l}$. For $m > 2n$ (19) and (20) can be used again. That all field equations are satisfied at all orders can be proved much as in the proof of Theorem 2, always proceeding in the direction of decreasing l for each fixed m.

A question left open by Theorem 3 is whether it can ever happen that any of the coefficients with $l > 0$ are non-zero. This is equivalent to the question whether $(\sigma^a{}_b)_{n,1}$ is ever non-zero. It follows from the proof of the theorem that this coefficient is uniquely determined by g^0_{ab}. In the case $n = 2$ the coefficient of interest vanishes due to the fact that the Ricci tensor of a two-dimensional metric is automatically traceless. For all even dimensions greater than two there are choices of g^0_{ab} for which $(\sigma^a{}_b)_{n,1}$ does not vanish. In fact this is the generic case. The coefficient of interest can be written as a polynomial expression in H^{-1}. If there were no logarithmic terms for a given choice of g^0_{ab} then all terms in this polynomial would have to vanish. The coefficient of H^{-n+1}, which is the most negative power of H occurring, is a non-zero constant times $\tilde{P}^a{}_b(\mathrm{tr}P)^{k-1}$, where $k = n/2$ and $\tilde{P}^a{}_b$ and $\mathrm{tr}P$ are the tracefree part and trace of the Ricci tensor of g^0_{ab}. There are only two ways in which this coefficient can vanish. Either the scalar curvature of g^0_{ab} vanishes identically or g^0_{ab} is an Einstein metric. A necessary condition for the absence of logarithmic terms has now been given but it is unlikely to be sufficient. The coefficients of other powers of H have to be taken into account in order to decide this issue.

In [18] the expansions obtained for vacuum spacetimes were extended to the case of a perfect fluid with pressure proportional to energy density. Formalizing these considerations leads to a theorem generalizing Theorem 1 above. The notation here is as follows: $\rho = T^{00}$, $j^a = T^{0a}$ and $S^{ab} = T^{ab}$. The proper energy density and pressure of the fluid are denoted by μ and p respectively, so that

$$T^{\alpha\beta} = (\mu + p)u^\alpha u^\beta + pg^{\alpha\beta} \tag{23}$$

The equation of state is taken to be $p = (\gamma - 1)\mu$ with $1 \leq \gamma < 2$. In the case with matter evolution and constraint quantities can be defined by

$$\tilde{E}^a{}_b = \partial_t\sigma^a{}_b - [\tilde{R}^a{}_b + (\mathrm{tr}k)\sigma^a{}_b - 8\pi\tilde{S}^a{}_b] \tag{24}$$

$$E = \partial_t(\mathrm{tr}k) - [R + (\mathrm{tr}k)^2 + 4\pi\mathrm{tr}S - 12\pi\rho - 3\Lambda] \tag{25}$$

$$C = R - k^{ab}k_{ab} + (\mathrm{tr}k)^2 - 16\pi\rho - 2\Lambda \tag{26}$$
$$C_a = \nabla_b k^b{}_a - \nabla_a(\mathrm{tr}k) - 8\pi j_a \tag{27}$$

so that their vanishing is equivalent to the Einstein equations. These satisfy the consistency conditions (8) and (9) as in the vacuum case. The components of the energy-momentum tensor can be expressed in terms of the fundamental fluid variables as follows:

$$\rho = \mu(1+\gamma|u|^2) \tag{28}$$
$$j^a = \gamma\mu(1+|u|^2)^{1/2}u^a \tag{29}$$
$$S^a{}_b = \mu[\gamma u^a u_b + (\gamma-1)\delta^a_b] \tag{30}$$

where $|u|^2 = g_{ab}u^a u^b$. The following relations will be useful:

$$\partial_t\rho - (\mathrm{tr}k)\rho - \frac{1}{3}(\mathrm{tr}k)\mathrm{tr}S = -\nabla_a j^a + \sigma^a{}_b S^b{}_a \tag{31}$$
$$\partial_t j^a - \frac{5}{3}(\mathrm{tr}k)j^a = -\nabla^b S^a{}_b + 2\sigma^a{}_b j^b \tag{32}$$

It is possible to express μ and u^a in terms of ρ and j^a. To see this note first that $|j| = \gamma\mu(1+|u|^2)^{1/2}|u|$ and that as a consequence:

$$|j|^2/\rho^2 = \gamma^2(1+|u|^2)|u|^2/(1+\gamma|u|^2)^2 \tag{33}$$

If $f(x) = \gamma^2x^2(1+x^2)(1+\gamma x^2)^{-2}$ then $f'(x) = 2\gamma^2x(1+\gamma x^2)^{-3}(1+(2-\gamma)x^2) > 0$. It follows that the mapping from the interval $[0,\infty)$ to the interval $[0,1)$ defined by f is invertible and $|u|^2$ can be expressed as a smooth function of $|j|^2/\rho^2$ for $\rho > 0$. Since μ can be expressed as a smooth function of ρ and $|u|^2$ it follows that it is a smooth function of ρ and j^a. Similarly the fact that u^a can be expressed as a smooth function of μ, $|u|^2$ and j^a implies that u^a is a smooth function of ρ and j^a.

Next Theorem 1 will be generalized to the case with perfect fluid. The solution is sought as a formal series where each tensor occurring is written as a sum of exponentials. The exponents are taken from an increasing sequence of real numbers $M = \{m_i\}$ which tends to infinity as $i \to \infty$. The solution is of the form

$$g_{ab} = \sum_{m_i\in M}(g_{ab})_{m_i}e^{-m_iHt} \tag{34}$$
$$\mu = \sum_{m_i\in M}(\mu)_{m_i}e^{-m_iHt}$$
$$u^a = \sum_{m_i\in M}(u^a)_{m_i}e^{-m_iHt}$$

Let integers k_1, k_2, k_3 and k_4 be defined as follows. For $\gamma \le 4/3$ we have $k_1 = 3\gamma$, $k_2 = 5-3\gamma$, $k_3 = 3\gamma$, $k_4 = 5$ while for $\gamma \ge 4/3$ we have $k_1 = 2\gamma/(2-\gamma)$, $k_2 = (6-4\gamma)/(2-\gamma)$, $k_3 = 4$ and $k_4 = 5$. In order to organize the coefficients it is useful to define the relative order $\tilde{m}_i$ of a coefficient of order m_i. For the quantities g_{ab}, $k^a{}_b$, μ, u^a, ρ and j^a these are defined by $\tilde{m}_i = m_i + 2$, $\tilde{m}_i = m_i$, $\tilde{m}_i = m_i - k_1$, $\tilde{m}_i = m_i - k_2$, $\tilde{m}_i = m_i - k_3$ and $\tilde{m}_i = m_i - k_4$ respectively.

Theorem 4 Let A_{ab} be a smooth three-dimensional Riemannian metric and B_{ab} a smooth symmetric tensor, μ_0 a smooth positive real-valued function and u_0^a a smooth vector field. Suppose that $A^{ab}B_{ab} = -(8\pi/3H^2)\mu_0$ for $\gamma = 1$, $A^{ab}B_{ab} = 0$ for $\gamma > 1$ and $\nabla^a B_{ab} = \nabla_b(A^{ac}B_{ac}) + (16\pi\gamma/3H)\mu_0 A_{bc}u_0^c$ where the covariant derivative is that associated to A_{ab}. If $\gamma > 4/3$ suppose furthermore that u^a is nowhere vanishing. Then there exists a unique formal power series solution of the Einstein-Euler equations with cosmological constant $\Lambda > 0$ and equation of state $p = (\gamma - 1)\rho$, $1 \le \gamma < 2$, of the form (34) with $(g_{ab})_{-2} = A_{ab}$, $(g_{ab})_1 = B_{ab}$, $(\mu)_{k_1} = \mu_0$ and $(u^a)_{k_2} = u_0^a$. The coefficients of the series are smooth. They satisfy $(\mu)_{m_i} = 0$ for $m_i < k_1$, $(u^a)_{m_i} = 0$ for $m_i < k_2$ and, except for $m_i = -2$, the coefficient $(g_{ab})_{m_i}$ vanishes for $m_i < 0$.

Proof Consider a formal series solution whose coefficients vanish in the ranges indicated in the statement of the theorem. With the given values for k_1 and k_2 the matter terms do not contribute to the equations for the coefficients of $k^a{}_b$ below order three and thus all statements made about these coefficients in the vacuum case can be taken over without change. This follows from the fact that ρ, $S^a{}_b$ and j_a are all $O(e^{-3Ht})$. The exponent in this estimate can be improved except in the case of ρ with $\gamma = 1$ and in the case of j_a with general γ.

The proof splits into several cases. Suppose first that $1 \le \gamma < 1/3$. Then $|u| = o(1)$ and so in leading order $\rho = \mu$, $j^a = \gamma\mu|u|u^a$ and $S^a{}_b = (\gamma - 1)\mu\delta^a_b$. Thus the following relations are obtained:

$$(m_i - 3\gamma)(\rho)_{m_i} = \ldots \quad (35)$$

$$(m_i - 5)(j^a)_{m_i} = \ldots \quad (36)$$

The terms not written out explicitly are lower order in the sense that they are combinations of terms of lower relative order than $\tilde{m}_i$. There is one subtlety involved in showing this. In the case $\gamma = 1$ the expression $\nabla^b S^a{}_b$ gives rise to a term which, looking at the exponents, is not lower order. However the coefficient of this term contains a factor $\gamma - 1$ and so the term vanishes for $\gamma = 1$. The Einstein equations give:

$$(3 - m_i)(\sigma^a{}_b)_{m_i} = \ldots \quad (37)$$

and

$$(6 - m_i)(\mathrm{tr}k)_{m_i} = -12\pi H^{-1}(\rho)_{m_i} + \ldots \quad (38)$$

The terms on the right hand side of the last two equations not written out explicitly are lower order. The one explicit term on the right hand side of the last equation is also lower order except in the case $\gamma = 1$. The energy-momentum quantities ρ and j^a are linked to the matter quantities μ and u^a by the relations

$$(\mu)_{m_i} = \rho_{m_i} + \ldots \quad (39)$$

$$(u^a)_{m_i} = \gamma^{-1}(\rho^{-1}j^a)_{m_i} + \ldots \quad (40)$$

Fix a value of $\tilde{m}_i$ and suppose that all coefficients with lower relative order have been determined. Consider the equations for $(\rho)_{m_i}$ and $(u^a)_{m_i}$. These coefficients are determined uniquely unless $\tilde{m}_i = 0$ and if $\tilde{m}_i < 0$ they vanish. When $\tilde{m}_i = 0$ they are determined by the conditions on $(\mu)_{k_1}$ and $(u^a)_{k_2}$ in the hypotheses of the theorem and the equations relating ρ and j^a to μ and u^a. The latter relations also fix the coefficients of μ and u^a of the given relative order when $\tilde{m}_i > 0$. Next consider the equations for $(\mathrm{tr}k)_{m_i}$ and $(k^a{}_b)_{m_i}$. By what has been said above we may assume that $m_i \geq 3$. The unique determination of the coefficients of the given relative order can be shown using the same procedure as in the vacuum case. The additional terms are either already of lower relative order, and hence known, or have been determined in the preceeding discussion of the matter equations. By induction on i it can be concluded that all coefficients are uniquely determined. The fact that all field equations are satisfied can be shown much as in the vacuum case since the compatibility conditions are identical.

Now consider the case $4/3 < \gamma < 2$. The assumption that u_0^a is nowhere vanishing implies in this case that $|u|^{-1} = o(1)$ and in leading order $\rho = \gamma\mu|u|^2$, $j^a = \gamma\mu|u|u^a$ and $S^a{}_b = \gamma\mu u^a u_b$. The following relations are obtained:

$$(m_i - 4)(\rho)_{m_i} = \ldots \tag{41}$$

$$(m_i - 5)(j^a)_{m_i} = \ldots \tag{42}$$

The Einstein equations give the same relations as in the previous case. For $4/3 < \gamma < 2$ the energy-momentum quantities ρ and j^a are linked to the matter quantities μ and u^a by the relations

$$(\mu)_{m_i} = (2-\gamma)^{-1}(\rho(1-|j|^2/\rho^2))_{m_i} + \ldots \tag{43}$$

$$(u^a)_{m_i} = ((2-\gamma)/\gamma)^{1/2}(\rho^{-1}(1-|j|^2/\rho^2)^{-1/2}j^a)_{m_i} + \ldots \tag{44}$$

Using these facts we can proceed as in the case $1 \leq \gamma < 4/3$.

Consider finally the case $\gamma = 4/3$ where $|u|$ tends to a finite limit, in general non-zero, as $t \to \infty$. The difference in comparison to the cases already treated is that the relations between ρ and j^a on the one hand and μ and u^a on the other hand cannot be inverted explicitly in leading order. However the fact, shown above, that the relevant mappings are invertible has an equivalent on the level of formal power series. For if f is a smooth function between open subsets of Euclidean spaces then $f(x+y)$ can be written formally in terms of a Taylor series about x. The resulting expression contains the derivatives of f evaluated at x multiplied by powers of y. If y is replaced by a formal power series without constant term then a well-defined formal power series for $f(x+y)$ is obtained. Thus the same method can be applied as in the previous cases, allowing the proof of the theorem to be completed.

A case which has been excluded in the above theorem is that where $\gamma > 4/3$ and u^a may vanish somewhere. In that case the kind of series which has been assumed in the theorem is not consistent. For if it is assumed that an expansion of this kind is possible this leads to different rates of decay for certain quantities, for instance μ, depending on whether $|u|$ does or does not vanish. As a consequence $\nabla_a\mu/\mu$ will be unbounded as t tends to infinity although μ is nowhere zero. This contradicts the assumptions which have been made. The situation is reminiscent of the spikes observed near the initial singularity in Gowdy spacetimes [17] and so we may speculate that in reality inhomogeneous features develop in μ so that the density contrast blows up as $t\to\infty$. This behaviour for $\gamma>4/3$ is not consistent with the usual picture in inflationary models where the density contrast remains bounded at late times. The issue deserves to be investigated further.

It is interesting to ask whether the expansions for a fluid presented here can be extended to the case of collisionless matter. If they can then the result probably resembles that for dust. Limited expansions in some special cases are already known [13], [20].

Note that the analysis of vacuum spacetimes in this section has a close analogue for Riemannian (i.e. positive definite) metrics. A solution of the Einstein equations with positive cosmological constant in the Lorentzian case corresponds to an Einstein metric with negative Einstein constant in the Riemannian case. The equations obtained for a positive definite metric are

$$-R-k^{ab}k_{ab}+(\mathrm{tr}k)^2 = -(n-1)K \quad (45)$$

$$\nabla_a k^a{}_b-\nabla_b(\mathrm{tr}k) = 0 \quad (46)$$

$$\partial_t k^a{}_b = -R^a{}_b+(\mathrm{tr}k)k^a{}_b+2K\delta^a{}_b \quad (47)$$

where K is the Einstein constant, i.e. the $n+1$-dimensional metric satisfies $R_{\alpha\beta}=Kg_{\alpha\beta}$. Asymptotic expansions for this case have been investigated in the literature on Riemannian geometry [3] and string theory [6].

3 From minimal to full asymptotics

In the last section consistent formal asymptotic expansions were exhibited for a number of problems. In this section it is shown that minimal information about the asymptotics implies the full expansions given in the last section. For simplicity we restrict consideration to the vacuum case. The following lemma will be used:

Lemma 1 Consider an equation of the form

$$\partial_t u+ku=\sum_{m,l} v_{m,l}t^l e^{-mt}+O(e^{-jt}) \quad (48)$$

for a vector-valued function $u(t)$, where $j\neq k$ and $m<j$ in the sum. Then there are coefficients $u_{m,l}$, $m<j$, such that

$$u=\sum_{m,l} u_{m,l}t^l e^{-mt}+O(e^{-jt}) \quad (49)$$

If (48) may be differentiated term by term with respect to t as often as desired the same is true of (49).

Proof Note first that $\partial_t(e^{kt}u)$ is equal to a sum of explicit terms with a term of order $e^{(k-j)t}$. Each of the explicit terms has an explicit primitive which is a sum of terms of the same general form and the same value of m but in general several values of l. Thus we can absorb these terms into the time derivative and write

$$\partial_t(e^{kt}(u-\sum_{m,l}u_{m,l}t^l e^{-mt}))=O(e^{(k-j)t}) \tag{50}$$

with $m<j$ in the sum. If $j<k$ we can integrate this relation directly to get the desired result. If $j>k$ then the expression which is differentiated with respect to time converges to a limit as $t\to\infty$, which can be called $u_{k,0}$. This gives the desired result in the latter case.

If the assumption on time derivatives is satisfied then $\partial_t u$ satisfies an equation of the same form as that satisfied by u. Hence $\partial_t u$ has an asymptotic expansion

$$\partial_t u=\sum_{m,l}w_{m,l}t^l e^{-mt}+O(e^{-jt}) \tag{51}$$

Integrating this from t_0 to t and using (49) gives

$$\sum_{m,l}\int_{t_0}^{t}w_{m,l}s^l e^{-ms}ds=C+\sum_{m,l}u_{m,l}t^l e^{-mt}+O(e^{-jt}) \tag{52}$$

for a constant C. It follows that the coefficients $w_{m,l}$ are obtained from $u_{m,l}$ by term by term differentiation. This process can be repeated for higher order derivatives with respect to t.

Remark If the quantities in (48) depend smoothly on a parameter and the equation may be differentiated term by term with respect to the parameter then the same is true for the solution.

Theorem 5 Let a solution of the vacuum Einstein equations with cosmological constant $\Lambda>0$ in $n+1$ dimensions be given in Gauss coordinates. Suppose that $e^{-2Ht}g_{ab}$, $e^{2Ht}g^{ab}$, $e^{2Ht}\sigma^a{}_b$ and their spatial derivatives of all orders are bounded. Then the solution has an asymptotic expansion of the form given in Theorem 3. The expansion remains valid when differentiated term by term to any order.

Proof The Hamiltonian constraint can be used to express tr$k$ in terms of the scalar curvature R, $\sigma^a{}_b$ and Λ, giving

$$\mathrm{tr}k=-\left[\frac{n}{n-1}\left(-R+\sigma^a{}_b\sigma^b{}_a\right)+n^2H^2\right]^{1/2} \tag{53}$$

It follows from the assumptions of the theorem that tr$k=-nH+O(e^{-2Ht})$ and that this relation may be differentiated term by term with respect to the spatial variables. Now

$$\partial_t(e^{-2Ht}g_{ab}) = -2e^{-2Ht}g_{ac}(k^c{}_b + H\delta^c_b) \tag{54}$$

The right hand side of this expression is $O(e^{-2Ht})$ and so there is some g^0_{ab} such that

$$e^{-2Ht}g_{ab} = g^0_{ab} + O(e^{-2Ht}) \tag{55}$$

and corresponding relations hold for spatial derivatives of all orders. Using the evolution equations it can be seen that these relations can also be differentiated repeatedly with respect to time. The proof now procedes by induction. The inductive hypothesis is as follows. There exist coefficients $(g_{ab})_{m,l}$ and $(k^a{}_b)_{m,l}$,$0 \leq m \leq M$ such that

$$g_{ab} = e^{2Ht}(\sum_{m=0}^{M}\sum_{l=0}^{L_m}(g_{ab})_{m-2,l}t^l e^{-mHt} + \bar{g}_{ab}) = [g_{ab}]_M + e^{2Ht}\bar{g}_{ab} \tag{56}$$

$$k^a{}_b = \sum_{m=0}^{M}\sum_{l=0}^{L_m}(k^a{}_b)_{m,l}t^l e^{-mHt} + \bar{k}^a{}_b = [k^a{}_b]_M + \bar{k}^a{}_b \tag{57}$$

where $\bar{g}_{ab}$ and $\bar{k}^a{}_b$ are $O(e^{-(M+\epsilon)Ht})$ and similar asymptotic expansions hold for all derivatives of these quantities. Here ϵ is a constant belonging to the interval $(0,1)$. The inductive hypothesis is satisfied for $M = 1$. If these expressions are substituted into the Einstein equations then the expansion coefficients written explicitly satisfy the same relations as in the analysis of formal power series solutions carried out above. It is convenient to write the evolution equations in the following form:

$$\partial_t\hat{g}_{ab} = -2\hat{g}_{ac}\sigma^c{}_b - (2/n)(\mathrm{tr}k + nH)\hat{g}_{ab} \tag{58}$$

$$\partial_t\sigma^a{}_b + nH\sigma^a{}_b = (\mathrm{tr}k + nH)\sigma^a{}_b + \tilde{R}^a{}_b \tag{59}$$

$$\partial_t(\mathrm{tr}k + nH) + 2nH(\mathrm{tr}k + nH) = (\mathrm{tr}k + nH)^2 + R \tag{60}$$

where $\hat{g}_{ab} = e^{-2Ht}g_{ab}$. Using the inductive hypothesis it follows that if each quantity Q in these equations is replaced by the corresponding quantity $[Q]_{M+1}$ then equality holds up to a remainder of order $e^{-(M+1+\epsilon)Ht}$ in (59) and (60). Using this information shows that the corresponding statement holds in (58) with a remainder of order $e^{-(M-1+\epsilon)Ht}$. Thus the quantities $[Q]_{M+1}$ satisfy a system of the type occurring in Lemma 1. It follows from that lemma that the inductive hypothesis is satisfied with M replaced by $M + 1$.

In [14] results similar to those of this section were obtained using a different coordinate system. The time coordinate used there satisfies the condition that that the lapse function is proportional to the inverse of the mean curvature of its level surfaces. This means that the foliation of level surfaces is a solution of the inverse mean curvature flow, a fact which raises serious doubts whether such coordinates exist in forever expanding cosmological spacetimes, as will now be explained. The inverse mean curvature flow for hypersurfaces is defined by the condition that a hypersurface flows with a speed equal to the inverse of its mean curvature in the normal direction. In the case of a Riemannian manifold it was used in the work of Huisken and Ilmanen [10] on the Penrose inequality. For spacelike hypersurfaces in a Lorentzian manifold it was studied in [9]. If a spacelike hypersurface with positive expansion (i.e., in the convention used here, with $\mathrm{tr}k < 0$) is given then there is a local solution of the inverse mean curvature flow in the contracting direction. Moreover, under reasonable assumptions on the nature of the singularity, there is a global solution. In the expanding direction, in contrast, the equation is backward parabolic and it is to be expected that there is no local solution for general initial data, i.e. for a general starting hypersurface. This is an analogue of the fact that the heat equation cannot be solved backwards in time.

4 Relations to conformal infinity

There is a relation between the expansions discussed in the last two sections and the concept of conformal infinity. In this section only the Einstein vacuum equations are considered. Define $T = H^{-1}e^{-Ht}$. Then spacetime metric corresponding to (18) becomes

$$(HT)^{-2}[-dT^2 + (g^0_{ab} + \sum_{m=0}^{\infty}\sum_{l=0}^{L_m}(g_{ab})_{m,l}(-1)^l H^{-l}(\log(HT))^l(HT)^m)] \qquad (61)$$

It is conformal to a metric which is non-degenerate at $T = 0$ and is written in Gauss coordinates. If there are no non-vanishing coefficients with $l > 0$ the conformal metric (or unphysical metric) is smooth at $T = 0$. This is for instance the case when n is odd.

In the case $n = 3$ Friedrich [4], [5] has used conformal techniques to prove results which, as shown in the following, imply that spacetimes evolving from initial data close to standard initial data for de Sitter space indeed have asymptotic expansions of the type presented in the last section. The method used, based on the conformal method, is only known to work in the case $n = 3$. The occurence of logarithms in the expansions for even values of n cast doubt on the possibility of implementing an analogous procedure in that case. There are also problems for $n = 3$ if matter is present. For conformally invariant matter fields the method can be used but for other types of matter, e.g. a perfect fluid with linear equation of state, there is no straightforward way of doing this. The non-integer powers occurring in the formal expansions for this case make the application of the method problematic. Note, however, that a similar problem has been overcome in the study of isotropic singularities [2].

Consider the de Sitter solution with a slicing by intrinsically flat hypersurfaces, as described in Section 2, with the slicing being given by the hypersurfaces of constant t. We may assume for convenience that the solution has been identified in a way which is periodic in the spatial coordinates. Consider initial data which is a small perturbation of the initial data induced by this model solution on a hypersurface of constant time. The smallness can be measured in the sense of uniform convergence of a function and its derivatives of all orders. Then, according to section 9 of [5], the perturbed solution has a Cauchy development which is asymptotically simple in the future. This means that the solution $g_{\alpha\beta}$ is conformal to a metric $\tilde{g}_{\alpha\beta}=\Omega^2 g_{\alpha\beta}$ with $\Omega>0$ in such a way that $g_{\alpha\beta}$ and Ω have smooth extensions through a hypersurface where Ω vanishes. We may choose coordinates in the unphysical metric in the following way. Set $\tilde{T}=\Omega$ and choose the spatial coordinates to be constant along the curves orthogonal to the hypersurfaces of constant Ω. In these coordinates the conformal metric takes the form

$$-H^{-2}\alpha^2 d\tilde{T}^2+\tilde{g}_{ab}dX^a dX^b \tag{62}$$

where α is a function of $\tilde{T}$ and X^a. The condition $-3\nabla_\alpha\Omega\nabla^\alpha\Omega=\Lambda$ (see Lemma 9.2 of [5]) implies that $\alpha=1$ for $\tilde{T}=0$. In order to compare this with the expansions in the previous sections we need to transform to Gauss coordinates with respect to the physical metric $g_{\alpha\beta}$. As a first step let $\tilde{T}=e^{-HT}$. Then the physical metric becomes

$$-\alpha^2 dT^2+g_{ab}dX^a dX^b \tag{63}$$

with $g_{ab}=e^{2HT}\tilde{g}_{ab}$. The following lemma shows that Gauss coordinates of a suitable kind can be introduced. The hypotheses make use of the following inequalities

$$|g_{ab}| \le Ce^{2HT} \tag{64}$$

$$|\alpha^{-1}|+|\Gamma^a_{bc}| \le C \tag{65}$$

$$|\alpha-1|+|\partial_T\alpha|+|\partial_a\alpha| \le Ce^{-HT} \tag{66}$$

$$|\tilde{k}^a{}_b|+|\mathrm{tr}k+3H|+|g^{ab}| \le Ce^{-2HT} \tag{67}$$

The metric (63) above satisfies inequalities of this type together with corresponding inequalities for spatial derivatives of all orders. In fact the estimates for $\tilde{k}^a{}_b$ and $\mathrm{tr}k+H$ are only obviously satisfied with the bound Ce^{-HT}. However this can be improved by using equations (59) and (60) at the end of the last section, or rather their equivalents in the presence of a non-trivial lapse function.

Lemma 2 Consider a metric of the form (63) on a time interval $[T_0,\infty)$ and assume that there is a constant $C>0$ such that the inequalities (64)-(67) are satisfied, together with the corresponding inequalities for spatial derivatives of all orders. Then for T_0 sufficiently large there exists a Gaussian coordinate system based on the hypersurface $T=T_0$ which is global in the future. The transformed metric satisfies the hypotheses of Theorem 5.

Proof To construct Gaussian coordinates it is necessary to analyse the equations of timelike geodesics. In 3+1 form these are

$$\frac{d^2T}{d\tau^2}+\alpha^{-1}\partial_T\alpha\left(\frac{dT}{d\tau}\right)^2+2\alpha^{-1}\nabla_a\alpha\frac{dX^a}{d\tau}\frac{dT}{d\tau}+\alpha^{-1}k_{ab}\frac{dX^a}{d\tau}\frac{dX^b}{d\tau}=0 \tag{68}$$

$$
\begin{aligned}
\frac{d^2X^a}{d\tau^2} &+ \alpha\nabla^a\alpha\left(\frac{dT}{d\tau}\right)^2 - \frac{2}{3}\alpha(\mathrm{tr}k)\frac{dX^a}{d\tau}\frac{dT}{d\tau} \\
&- 2\alpha\tilde{k}^a{}_b\frac{dX^b}{d\tau}\frac{dT}{d\tau} + \Gamma^a_{bc}\frac{dX^b}{d\tau}\frac{dX^c}{d\tau} = 0
\end{aligned}
\tag{69}
$$

It is helpful for the following analysis to rewrite one of the terms:

$$
\begin{aligned}
-\frac{2}{3}\alpha(\mathrm{tr}k)\frac{dX^a}{d\tau}\frac{dT}{d\tau} &= 2H\frac{dX^a}{d\tau} + 2H\frac{dX^a}{d\tau}\left(\frac{dT}{d\tau}-1\right) \\
&- \frac{2}{3}(\mathrm{tr}k+3H)\frac{dX^a}{d\tau}\frac{dT}{d\tau} - \frac{2}{3}(\alpha-1)(\mathrm{tr}k)\frac{dX^a}{d\tau}\frac{dT}{d\tau}
\end{aligned}
\tag{70}
$$

These equations are to be solved for functions $T(\tau,x^b)$ and $X^a(\tau,x^b)$ with initial values $T=T_0$, $dT/d\tau=1$, $X^a=x^a$ and $dX^a/d\tau=0$ at $\tau=T_0$. Strictly speaking Gaussian coordinates based on $T=T_0$ would differ from this by a time translation by T_0 but it is convenient here to work with this slight modification. Consider now a solution of these equations on an interval $[T_0,\tau^*)$ and suppose for later convenience that $T_0\ge 0$. There is a $\tau^*>T_0$ for which a solution does exist. We assume that on this interval $|dX^a/d\tau|\le Ce^{-2H\tau}$ and that $|dT/d\tau-1|<\epsilon$ for some $\epsilon\in(0,1/3)$. For given C and ϵ there exists an interval of this kind. On this interval $e^{-T}\le e^{-\epsilon\tau_0}e^{-(1-\epsilon)\tau}$, $e^{T-\tau}\le e^{\epsilon(\tau-\tau_0)}$ and inequalities of the following form hold, where C' is a positive constant depending only on C and ϵ.

$$
\frac{d^2T}{d\tau^2} = f(\tau),\quad |f(\tau)|\le C'e^{-(1-\epsilon)H\tau} \tag{71}
$$

$$
\frac{d^2X^a}{d\tau^2}+2H\frac{dX^a}{d\tau} = g(\tau),\quad |g(\tau)|\le C'e^{-2H\tau} \tag{72}
$$

It follows from the first of these that

$$
|dT/d\tau-1|\le C'e^{-(1-\epsilon)HT_0} \tag{73}
$$

For T_0 large enough this strictly improves on the estimate originally assumed for $dT/d\tau-1$. For small ϵ the quantities $dX^a/d\tau$ can be seen to decay exponentially with an exponent which is as close as desired to -2. The fact that we are dealing with timelike geodesics parametrized by proper time leads to the relation

$$
-1 = -\alpha^2\left(\frac{dT}{d\tau}\right)^2 + g_{ab}\frac{dX^a}{d\tau}\frac{dX^b}{d\tau} \tag{74}
$$

This implies that $|dT/d\tau-1|=O(e^{-HT})$. Putting this back into the evolution equation for $dX^a/d\tau$ shows that it is $O(e^{-2H\tau})$. By choosing T_0 large enough the decay estimate for this quantity is recovered and in fact strengthened. Consideration of the longest time interval on which the original inequality holds shows that $\tau^*=\infty$. The estimates we have derived up to now hold globally. The estimate for $dT/d\tau$ obtained above implies that there are positive constants

C_1 and C_2 such that $C_1\tau \le T \le C_2\tau$. Hence in estimates we can replace e^{-T} by $e^{-\tau}$ if desired.

Next we would like to obtain corresponding estimates for the spatial derivatives of $dT/d\tau$ and $dX^a/d\tau$ of all orders. Consider the result of differentiating the geodesic equations with respect to the spatial variables. This leads to estimates of the form

$$\frac{d}{d\tau}\left(\frac{\partial^2 T}{\partial\tau\partial x^a}\right) = f_a(\tau), \quad |f_a(\tau)| \le C' e^{-H\tau} \tag{75}$$

$$\frac{d}{d\tau}\left(\frac{\partial^2 X^a}{\partial\tau\partial x^c}\right) + 2H\frac{\partial^2 X^a}{\partial\tau\partial x^c} = g^a_c(\tau), \quad |g^a_c(\tau)| \le C' e^{-2H\tau} \tag{76}$$

This allows us to show that the first order spatial derivatives of the key quantities satisfy the estimates analogous to those satisfied by the quantities themselves. The same argument can be applied to estimate spatial derivatives of any order inductively. Now all the desired information about existence and decay of T and X^a has been obtained. It remains to show that they form a coordinate system. This follows from the fact that the initial values of $\partial T/\partial\tau$, $\partial X^a/\partial\tau$ and $\partial X^a/\partial x^b$ are one, zero and δ^a_b respectively and the exponential decay of their time derivatives which has already been proved. This completes the proof of the lemma.

Combining Lemma 2 and Theorem 5 shows that the spacetimes constructed by Friedrich admit global Gaussian coordinates in which they have an asymptotic expansion of the form of (10). Hence any initial data close to that for de Sitter on a flat hypersurface evolves into a solution having an asymptotic expansion of the form given by Starobinsky.

5 The spatially homogeneous case

This section is concerned with spatially homogeneous solutions of the vacuum Einstein equations in $n+1$ dimensions with positive cosmological constant. It is assumed that the spatial homogeneity is defined by a Lie group G, supposed simply connected, which acts simply transitively. We restrict to spacetimes such that all left invariant Riemannian metrics on G have non-positive scalar curvature. In the case $n = 3$ this corresponds to Bianchi types I to VIII. Information on the case $n = 4$ can be found in [8]. It will be shown that the spacetimes of the type just specified have asymptotic expansions with all the properties of the formal expansions in Theorem 3.

A spatially homogeneous spacetime of the type being considered can be written in the form

$$-dt^2 + g_{ij}(t)e^i \otimes e^j \tag{77}$$

where $\{e^i\}$ is a left invariant frame on the Lie group G. Basic information about the asymptotics of these spacetimes in 3+1 dimensions are given by Wald's theorem [21], which provides information on the behaviour of the second fundamental form as $t \to \infty$. This can easily be generalized to the present situation.

Using the condition on the sign of R, the Hamiltonian constraint implies that on any interval where a solution exists $(\mathrm{tr}k)^2 \geq \frac{2n\Lambda}{n-1} = (nH)^2$. Combining the Hamiltonian constraint with the evolution equation for $\mathrm{tr}k$ gives

$$\partial_t(\mathrm{tr}k) \geq \frac{1}{n}(\mathrm{tr}k)^2 - \frac{2}{n-1}\Lambda \tag{78}$$

In particular $\mathrm{tr}k$ is non-decreasing. These facts together show that $\mathrm{tr}k$ is bounded. Now it will be shown that $\mathrm{tr}k \to -nH$ as $t \to \infty$. For

$$\begin{aligned} \partial_t(\mathrm{tr}k + nH) &\geq \frac{1}{n}(-\mathrm{tr}k + nH)(-\mathrm{tr}k - nH) && (79) \\ &\geq -2H(\mathrm{tr}k + nH) && (80) \end{aligned}$$

It follows that $\mathrm{tr}k = -nH + O(e^{-2Ht})$. Using the Hamiltonian constraint then gives $\sigma^i{}_j\sigma^j{}_i = O(e^{-2Ht})$. This bound can be used to get information on σ_{ij} as in [15]. Then it is possible to proceed exactly as in the proof of Proposition 2 in [13] to show that $e^{-2Ht}g_{ij}$, $e^{2Ht}g^{ij}$ and $e^{Ht}\sigma^i{}_j$ are bounded. Then equation (58) can be used as in the previous section to improve the last statement to the boundedness of $e^{2Ht}\sigma^i{}_j$. The fact that g_{ij}, g^{ij} and k_{ij} are bounded on any finite time interval implies that the solution exists globally in time.

We are now in a situation very similar to that of Theorem 5. However the estimates we have are expressed in term of frame components. Choosing a coordinate system on some subset of the Lie group G with compact closure will give us uniform asymptotic expansions for the components in that coordinate system. Conversely uniform asymptotic expansions for the components in a coordinate system of this type give corresponding asymptotic expansions for the frame components. In this case we will say that the asymptotic expansions are locally uniform. If an expansion of this type holds for a given quantity it also holds for all spatial derivatives in the coordinate representation.

The proof of Theorem 5 uses only arguments which are pointwise in space and so it generalizes immediately to the case of locally uniform asymptotic expansions. It can be concluded that for a spatially homogeneous spacetime of the type under consideration locally uniform asymptotic expansions of generalized Starobinsky type are obtained. Restricting to a coordinate domain with compact closure uniform asymptotic expansions are obtained. In general these expansions will contain logarithmic terms. Consider for instance the case of the Lie group $H \times \mathbf{R}$ where H is a three-dimensional Lie group of Bianchi type other than IX. Let the spacetime be such that the spatial metric at each time is the product of a metric on H with one on $\mathbf{R}$. This is consistent with the constraint equations. For instance the initial data can be chosen to be the product of data on H with trivial data on $\mathbf{R}$. The data are invariant under reflection in $\mathbf{R}$ and this property is inherited by the solutions. Suppose that H admits no metric of vanishing scalar curvature. This is the case for every Bianchi type except I and VII_0. Then the metric g^0_{ab} corresponding to this solution has non-vanishing scalar curvature and is not an Einstein metric. Hence logarithmic terms are unavoidable.

6 Fuchsian analysis

Fuchsian systems are a class of singular equations which can be used to prove the existence of solutions of certain partial differential equations with given asymptotics [11], [12], [16]. It will be shown that Fuchsian methods allow the construction of solutions of the vacuum Einstein equations with positive cosmological constant in any number of dimensions which have asymptotic expansions of the type given in Section 2 and depend on the same number of free functions as the general solution.

Before coming to the specific problem of interest here some general facts about Fuchsian equations will be recalled. The form of the equations is

$$t\partial u/\partial t + N(x)u = tf(t, x, u, u_x) \tag{81}$$

Here x denotes the spatial coordinates collectively and u_x the spatial derivatives of the unknown $u(t, x)$. The matrix N and the function f are required to satisfy certain regularity conditions and N is required to satisfy a positivity condition. There are forms of the regularity condition adapted to smooth and to analytic functions. The version adapted to analytic fucntions will be used in the following since it is the one where the most powerful theorems are available. For the precise definition of regularity see [1], where a corresponding definition of regularity of solutions is also given. Roughly speaking, regularity means that the functions concerned are continuous in t and analytic in x and vanish in a suitable way as $t \to 0$. Consider now an ansatz of the form

$$u(t, x) = \sum_{m=0}^{\infty} \sum_{l=0}^{L_m} u_{m,l} t^m (\log t)^l \tag{82}$$

By analogy with what was done in Section 2 we can ask whether the equation (81) has a formal series solution of this kind. Suppose that this is the case. Fix $M \geq 0$. Then there exist coefficients $u_{m,l}$ such that $\bar{u} = u - \sum_0^M \sum_0^{L_m} u_{m,l} t^m (\log t)^l$ satisfies

$$t\partial \bar{u}/\partial t + N(x)\bar{u} = t^{M+\epsilon} f_M(\bar{u}) \tag{83}$$

for a regular function f_M and a constant $\epsilon > 0$ together with the corresponding relations obtained by differentiating term by term with respect to the spatial coordinates any number of times. Let $v = t^{-M}\bar{u}$. Then

$$t\partial v/\partial t + (N(x) + MI)v = t^{\epsilon} g(t, x, v, v_x) \tag{84}$$

for a regular function g. Introducing t^{ϵ} as a new time variable, we obtain an equation of the form (81). If we assume that $N(x)$ is bounded then by choosing M large enough it can be ensured that the matrix $N(x)+MI$ is positive definite. Assuming that f and N are regular in the analytic sense the existence theorem of [1] implies the existence of a unique regular solution v vanishing at $t = 0$. Expressing u in terms of v gives a solution of the original equation which has the given asymptotic expansion up to order M.

Consider now the slightly more general equation

$$t\partial u/\partial t + N(x)u = tf(t,x,u,u_x) + h(u) \tag{85}$$

In order to have a consistent formal power series solution suppose that for some functions $u_{0,0}$ and $u_{1,0}$ we have $Nu_{0,0} = h(u_{0,0})$ and

$$(N + I - Dh(u_{0,0}))u_{1,0} = f(0, u_{0,0}) \tag{86}$$

Here Dh denotes the derivative of h as a map between Euclidean spaces. Suppose further that the equation admits a formal power series solution with coefficients $u_{0,0}$ and $u_{1,0}$ and $L_0 = L_1 = 0$. If these conditions hold then u satisfies the original equation if $v = u - u_{0,0} - u_{1,0}t$ satisfies a Fuchsian system and vanishes at the origin. Thus an existence theorem is obtained.

To make contact with the Einstein equations we start with the equations (58)-(60) and set $\tau = e^{-Ht}$. Then an equation of the form (85) is obtained, with $u = (\hat{g}_{ab}, \sigma^a{}_b, \mathrm{tr}k + nH)$. If it is assumed that the variables $\sigma^a{}_b$ and $(\mathrm{tr}k + nH)$ vanish at $\tau = 0$ then the consistency conditions on $u_{0,0}$ and $u_{1,0}$ are satisfied. The fact that consistent formal expansions were shown to exist in Section 2 allows the above procedure to be carried through. If the data A_{ab} and B_{ab} are chosen to be analytic then this gives an existence theorem for the Einstein evolution equations with A_{ab} and B_{ab} prescribed as in Theorem 3. In this context it is important to note that if A_{ab} and B_{ab} are analytic all the coefficients in the formal expansions whose existence is asserted in Theorem 3 are also analytic. In order to see that a solution of the Einstein equations is obtained it suffices to show that the constraint equations are satisfied. Note that it follows from the results of Section 2 that the constraint quantities vanish to all orders at $\tau = 0$ but since the solution is not analytic at $\tau = 0$ this does not suffice to conclude that the constraint quantities vanish everywhere. To see that they do we need to write the consistency conditions (8) and (9) in Fuchsian form. Using the fact that the Einstein evolution equations are satisfied, and introducing $\tilde{C}_a = e^{-tH}C_a$, these equations can be written as

$$\begin{aligned} \partial_t C + 2HC &= 2(\mathrm{tr}k + H)C - 2e^{Ht}\nabla^a C_a && (87)\\ \partial_t \tilde{C}_a + 2H\tilde{C}_a &= (\mathrm{tr}k + H)\tilde{C}_a - (1/2)e^{-Ht}\nabla_a \tilde{C} && (88) \end{aligned}$$

Setting $\tau = e^{-Ht}$ gives a system of the form (81) and since C and $\tilde{C}_a$ tend to zero as $\tau \to 0$ both of these quanitities vanish as a consequence of the uniqueness theorem for Fuchsian systems and the constraints are satisfied. The solution of the Einstein equations has an asymptotic expansion of the form given in Theorem 3 truncated at any given finite order. Applying Theorem 5 shows that this solution has an asymptotic expansion of this form to all orders. The results obtained can be summed up as follows:

Theorem 6 Let A_{ab} be an analytic n-dimensional Riemannian metric and B_{ab} an analytic symmetric tensor which satisfies $A^{ab}B_{ab} = Z(A)$ and $\nabla^a B_{ab} = \tilde{Z}_b(A)$, where the covariant derivative is that associated to A. Then there exists an analytic solution of the vacuum Einstein equations with an asymptotic expansion of the form (18) with $g^0_{ab} = A_{ab}$ and $(g_{ab})_{n-2,0} = B_{ab}$. The expansion may be differentiated term by term with respect to the spatial variables as often as desired.

7 The wave equation on de Sitter spacetime

In Section 4 it was shown that initial data for the vacuum Einstein equations in 3+1 dimensions close to that for de Sitter space evolve to give a spacetime with asymptotics of Starobinsky type. It has not yet proved possible to obtain the analogous statement in higher dimensions. What is missing are suitable energy estimates. In this section it will be shown how a simpler model problem can be treated. This is the case of the wave equation $\nabla_\alpha\nabla^\alpha\phi = 0$ on (the higher dimensional analogue of) de Sitter space. The spacetime metric in this case is

$$ds^2 = -dt^2 + e^{2Ht}((dx^1)^2 + \ldots + (dx^n)^2) \tag{89}$$

Written out explicitly in coordinates the wave equation takes the form:

$$\partial_t^2\phi + nH\partial_t\phi = e^{-2Ht}\Delta\phi \tag{90}$$

where Δ is the Laplacian of the flat metric.

Consider the ansatz for formal solutions of the equations

$$\sum_{m=0}^{\infty}(A_m(x)e^{-mHt} + B_m(x)te^{-mHt}) \tag{91}$$

Substituting this into the equation and comparing coefficients gives

$$m(m-n)H^2A_m - (2m-n)HB_m = \Delta A_{m-2} \tag{92}$$

$$m(m-n)H^2B_m = \Delta B_{m-2} \tag{93}$$

For any n it is true that $B_0 = A_1 = B_1 = 0$. In the case that n is odd assume that the coefficients B_m vanish. Then $\Delta A_{m-2} = H^2m(m-n)A_m$ for all $m \geq 2$. Then it follows from $A_1 = 0$ that $A_{2k+1} = 0$ for all integers k with $2k+1 < n$. The coefficients A_{2k+1} with $2k+1 > n$ are determined by A_n. The coefficients A_{2k} are determined by A_0. There are no further relations to be satisfied and so A_0 and A_n parametrize the general solution. If the coefficients B_m are not assumed to be zero it can be shown that they must vanish for n odd. For n even we have $A_{2k+1} = B_{2k+1} = 0$ for every positive integer k. If $2k < n$ then $B_{2k} = 0$. For $2k < n$ the coefficients A_{2k} are determined successively by A_0. Also B_n is determined by A_0. Then A_{2k} and B_{2k} are determined for all k with $2k > n$ in terms of the coefficients already determined. Thus the general solution can be parametrized by A_0 and A_n, just as in the case n odd. The difference is that the series obtained contains terms which are multiples of te^{-mHt}.

Let $E = \int(\partial_t\phi)^2 + e^{-2Ht}|\nabla\phi|^2$. Differentiating with respect to t and integrating by parts gives the relation $dE/dt = -2HE$. We can differentiate the equation through with respect to a spatial coordinate and repeat the argument. This shows that all Sobolev norms of $e^{Ht}\partial_t\phi$ and $\nabla\phi$ are bounded. By the Sobolev embedding theorem they and all their spatial derivatives satisfy corresponding pointwise bounds. Thus the spatial derivatives of ϕ are bounded while its time derivative decays like e^{-Ht}. As a consequence $\phi(t,x)=\phi_0(x)+O(e^{-Ht})$ for some function ϕ_0. Comparing with the formal solutions already obtained we see that these estimates are not likely to be sharp. The equation for ϕ is equivalent to the system

$$\partial_t\phi = \psi \tag{94}$$

$$\partial_t\psi + nH\psi = e^{-2Ht}\Delta\phi \tag{95}$$

Starting with the basic information on the asymptotic behaviour of ϕ we already have the method of proof of Theorem 5 can be applied to this system. The result is that any solution has an asymptotic expansion of the type derived on a formal level above.

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