Energy Conservation as the Basis of Relativistic Mechanics. II*

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From the relativity principle and the conservation of energy in particle collisions we deduce the form of the energy function, and the conservation of inertial mass and three-momentum. We show that the arguments are parallel under Einsteinian and Galilean kinematics.

A PAPER with the above title was recently published in this Journal by two of the authors.[1] In the present paper a modified argument is given which represents a significant improvement in brevity and precision. As the argument now stands, it would appear to be the simplest and pedagogically most satisfactory approach to (Einsteinian) relativistic mechanics suggested so far. For this reason, rather than merely communicate the changes from the previous paper, we thought it best to make the present account self-contained, especially since the overlap is slight. At the same time we have made brief mention of the parallel development of Newtonian mechanics. For a historical introduction and bibliography we still refer the reader to the earlier paper.

Our notation is the following. (It differs slightly from that of Ref. 1.) The magnitude of the 3-velocity $\mathbf{u}_F$ of a particle relative to an inertial frame F is denoted by u_F (the speed). The 4-velocity corresponding to any $\mathbf{u}$ is denoted by $\mathbf{U}$, and $\mathbf{U}\cdot\mathbf{V}$ means the Minkowskian scalar product $c^2U^4V^4-U^1V^1-U^2V^2-U^3V^3$ of two 4-vectors $\mathbf{U}=(U^1,U^2,U^3,U^4)$ and $\mathbf{V}=(V^1,V^2,V^3,V^4)$. The internal state of a particle, comprising all such properties as material constitution, spin, electric charge, temperature, etc., which can be measured in a rest frame of the particle, is indicated by a collective symbol $\mathcal{S}$. In order to divest $\mathcal{S}$ of its directional properties, we define *scalar* states S as equivalence classes of states $\mathcal{S}$, such that distinct $\mathcal{S}$ belong to the same S if they have the same description in suitably oriented rest frames.—Greek *superscripts* distinguish the states, velocities, etc., of different particles.

The mechanics to be discussed is that of particle collisions. We assume that the particles are "free," i.e., that their motions are independent of each other except during collisions, and that before and after collision each particle moves uniformly in accordance with the law of inertia (which is already incorporated into special relativity in the definition of inertial frames).

We make the following assumptions.

Assumption (I): There exists a real-valued universal energy function $E(u,S)$, where u ranges over-all speeds and S over-all scalar states, such that in any collision the energy sum relative to any inertial frame F is conserved:

$$\sum_{\alpha=1}^{n} E(u_F{}^\alpha,S^\alpha)=\sum_{\alpha=n+1}^{n+m} E(u_F{}^\alpha,S^\alpha) \tag{1}$$

(the particles numbered from 1 to n go into the collision, and those numbered from $n+1$ to $n+m$ come out of it). For any fixed S, $E(u,S)$ is a continuous nonconstant function of u.

The last property distinguishes an energy function from an "internal charge," i.e., a function of S alone whose sum is conserved in any collision. Since such an internal charge can be added to E without invalidating (1), it is clear that (1) does not define E uniquely.

A change of scalar state which occurs in a free particle without external influence can be considered as a one-into-one "collision." By the law of inertia such a change does not affect u, and by (1) it does not affect E. Hence, for the pur-

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[1] R. Penrose and W. Rindler, Am. J. Phys. **33**, 55 (1965).

poses of our analysis, such variable states can be considered as single states.

Assumption (II): The sum of the energies of two identical particles approaching each other with equal speeds u along any straight line in an inertial frame F is independent of this line, even relative to any other frame F'.

This is the analog, for the simplest of all 2-particle systems, of the direction-independence assumed for the energy of a single particle. Although this assumption cannot be deduced from Assumption (I), it is in accordance with the general energy principle. For we may suppose that the particles transfer their energy to a symmetric system at rest in F whose gain in energy relative to F must be the same in all cases; thus the gain relative to F' must also be the same.

Assumptions (I) and (II) can be used in conjunction with either Galilean or Einsteinian kinematics. We first consider the Einsteinian case.

For convenience, we introduce instead of u a new variable, namely the Lorentz factor

$$\gamma=\gamma(u)=(1-u^2/c^2)^{-\frac{1}{2}}; \qquad (2)$$

and we temporarily write $E(u,S)\equiv E[\gamma]$ when only one scalar state is under discussion. We recall that the 4-velocity associated with a 3-velocity $\mathbf{u}$ is given by $\mathbf{U}=\gamma(u)(\mathbf{u},c)$; hence the 4-velocity of an inertial observer relative to his own rest-frame is $\mathbf{V}=(\mathbf{0},c)$. Thus

$$\gamma=\mathbf{V}\cdot\mathbf{U}/c^2, \qquad (3)$$

i.e., the Lorentz factor of a particle relative to any inertial observer is given by the scalar product of the two corresponding 4-velocities divided by c^2. And this product, being invariant, can be evaluated in any inertial frame.

Now consider two particle systems as in Assumption (II), along the x and y axes, respectively. The components of the four relevant 4-velocities with respect to F are

$$\gamma(\pm u,\,0,\,0,\,c);\quad \gamma(0,\,\pm u,\,0,\,c);\quad [\gamma=\gamma(u)]. \qquad (4)$$

If F' moves with speed u' along the x axis of F, its 4-velocity components with respect to F are

$$\gamma'(u',0,0,c),\quad [\gamma'=\gamma(u')]. \qquad (5)$$

Hence, the various Lorentz factors of the particles relative to F' are given by the scalar products (divided by c^2) of (5) with the 4-velocities in (4):

$$\gamma_1=\gamma\gamma'(1+uu'/c^2),\quad \gamma_2=\gamma\gamma'(1-uu'/c^2),$$
$$\gamma_3=\gamma_4=\gamma\gamma'=\tfrac{1}{2}(\gamma_1+\gamma_2). \qquad (6)$$

Equating the energies of the two systems in F' in accordance with Assumption (II), we obtain the functional equation

$$E[\gamma_1]+E[\gamma_2]=2E[\tfrac{1}{2}(\gamma_1+\gamma_2)], \qquad (7)$$

valid for all values $1\leq\gamma_1,\ \gamma_2<\infty$ since u and u' can be varied independently. In words, this equation reads: "the average of E at γ_1 and γ_2 equals E at the average of γ_1 and γ_2." In the usual way (continued halving and appeal to the continuity of E) it follows that E must be a linear function of γ

$$E=c^2m\gamma+q, \qquad (8)$$

where m and q are functions of S (the factor c^2 being introduced for later convenience) and, because of the rider in Assumption (I),

$$m(S)\neq 0\quad \text{for any } S.\ ^{(2)} \qquad (9)$$

Now consider an arbitrary collision and let the 4-velocities of the particles involved be $\mathbf{U}^\alpha$. If $\mathbf{V}$ is the 4-velocity of an arbitrary inertial frame F (i.e., its "time axis") then according to (8) and (3) the energy sum relative to F can be written as

$$\mathbf{V}\cdot\sum_\alpha m^\alpha\mathbf{U}^\alpha+\sum_\alpha q^\alpha. \qquad (10)$$

Since this expression is conserved for each 4-velocity $\mathbf{V}$ it follows that $\sum m^\alpha\mathbf{U}^\alpha$ and $\sum q^\alpha$ are *separately* conserved. [Take, for example, $\mathbf{V}$ as in (5) and allow u' to vary. Then the conservation of (10) implies that of $\sum mU^1$, $\sum mU^4$, and $\sum q$; etc.] Note that q has therefore the character of an internal charge and mU^4 that of an energy function. We infer

Theorem (I): Under Einsteinian kinematics, Assumptions (I) and (II) are equivalent to the postulate that there exists a nonvanishing func-

[2] It would be interesting to know what further assumptions are sufficient within a classical collision theory to imply that m must have the same sign for all particles.

tion m of S such that the 4-vector sum

$$\sum_\alpha m^\alpha \mathbf{U}^\alpha \tag{11}$$

is conserved in all collisions.

We can now define the proper mass of a particle as the scalar m, and its 4-momentum as the 4-vector $m\mathbf{U}$, provided that m is unique (up to a universal factor corresponding to a change of unit). The conservation of 4-momentum, the central law of relativistic collision mechanics, would then be established.

Without attempting to give the most general uniqueness argument for m, we only state one simple assumption which is sufficient: Any two particles P and P' can be made either to collide and stick together upon impact; or to collide so that two particles again emerge and three of the four 4-velocities involved are linearly independent; or a finite number of particles $P_1, P_2, \cdots, P_n$ can be found such that each of the pairs P,P_1; P_1,P_2; $\cdots$; P_n,P' can undergo such a collision. For in any such collision the (observable) 4-velocities determine the *ratios* of the m uniquely. If there exist particles which can participate only in more complicated (e.g., 5-particle) collisions, the above assumption and argument would have to be somewhat extended.

The 3-momentum of a particle relative to a frame F can now be defined in the usual way as $m\gamma\mathbf{u}_F$ and the inertial mass as $m\gamma$, and both of these are conserved in any frame (the first being the spatial component and the second $1/c$ times the temporal component of the 4-momentum.)

Furthermore, $E=c^2m\gamma$ is seen to be *an* energy function satisfying Assumptions (I) and (II). This is evidently the most natural choice, since the addition of a charge-like quantity q as in (8) produces a hybrid entity, $c^2m\gamma$ transforming as the last component of a 4-vector ($cm\mathbf{U}$) and q as a scalar. On the other hand, it appears to be outside the scope of special relativity to prove that $c^2m\gamma$ is the *available* (as distinct from *conventional*) energy of a mass m. If, for example, the elementary particles were indestructible, the available energy of a macroscopic particle would be $c^2m\gamma+q$, where q is $-c^2$ times the sum of the rest masses of the constituent elementary particles. However, in general relativity, a kind of proof can be given by lowering—in imagination—a particle into the "Schwarzschild radius" of a mass point.

The basis of special relativistic mechanics is now established, and we need not discuss it further.

We finally formulate briefly the corresponding argument if Galilean kinematics is adopted. It is then convenient to use u^2 as a new variable instead of u; in place of (6) we obtain

$$u_1^2=(u-u')^2,\quad u_2^2=(u+u')^2,$$
$$u_3^2=u_4^2=u^2+u'^2=(u_1^2+u_2^2)/2, \tag{6'}$$

and in place of (7),

$$E[u_1^2]+E[u_2^2]=2E[\tfrac{1}{2}(u_1^2+u_2^2)]. \tag{7'}$$

Except for notation, this is identical with (7). Hence

$$E=\tfrac{1}{2}mu^2+E_0, \tag{8'}$$

where $m(\neq 0)$ and E_0 are functions of S, and the factor $\frac{1}{2}$ is included so that (8) and (8′) can coincide in the limit of small velocities. Instead of (10), the Galilean relativity principle leads to the conservation of

$$\sum \tfrac{1}{2}m^\alpha(\mathbf{u}^\alpha+\mathbf{u}')^2+\sum E_0{}^\alpha\equiv\sum(\tfrac{1}{2}m^\alpha u^{\alpha^2}+E_0)$$
$$+\mathbf{u}'\cdot\sum m^\alpha\mathbf{u}^\alpha+\tfrac{1}{2}u'^2\sum m^\alpha \tag{10'}$$

for each $\mathbf{u}'$ and, consequently, to the conservation of the quantities $\sum m^\alpha\mathbf{u}^\alpha$ and $\sum m^\alpha$, *separately.* A theorem analogous to Theorem (I) can therefore be formulated. It is also obvious that the analog of the argument for the uniqueness of m holds here.

The development (6′)–(10′) is essentially a "limit" of the special relativistic case, for small velocities. This becomes clear when it is remembered that $\gamma(u)=1+\frac{1}{2}u^2/c^2+O(1/c^4)$.

Thus in the Galilean as well as in the Einsteinian case, an energy conservation law in conjunction with the relativity principle implies a 3-momentum conservation law and the law of conservation of inertial mass. In the Einsteinian case, however, the energy function possesses a natural zero point whereas, since the two components of E in (8′) are not separately conserved, no such zero point exists in the Galilean case. In this respect, which has a deep physical significance, Einstein's theory is more determinate than its predecessor.