Hidden Symmetries in five-dimensional Supergravity

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Abstract

This thesis is concerned with the study of hidden symmetries in supergravity, which play an important role in the present picture of supergravity and string theory. Concretely, the appearance of a hidden $G_{2(+2)}/SO(4)$ symmetry is studied in the dimensional reduction of $d = 5, \mathcal{N} = 2$ supergravity to three dimensions – a parallel model to the more famous $E_{8(+8)}/SO(16)$ case in eleven-dimensional supergravity. Extending previous partial results for the bosonic part, I give a derivation that includes fermionic terms. This sheds new light on the appearance of the local hidden symmetry $SO(4)$ in the reduction, and shows up an unusual feature which follows from an analysis of the R-symmetry associated with $\mathcal{N} = 4$ supergravity and of the supersymmetry variations, and which has no parallel in the eleven-dimensional case: The emergence of an additional $SO(3)$ as part of the enhanced local symmetry, invisible in the dimensional reduction of the gravitino, and corresponding to the fact that, of the $SO(4)$ used in the coset model, only the diagonal $SO(3)$ is visible immediately upon dimensional reduction. The uncovering of the hidden symmetries proceeds via the construction of the proper coset gravity in three dimensions, and matching it with the Lagrangian obtained from the reduction.

Zusammenfassung

Thema dieser Arbeit sind die versteckten Symmetrien der Supergravitation, die im gegenwärtigen Verständnis des Wechselspiels von Stringtheorien und Supergravitation eine wichtige Rolle spielen. Konkret gilt die Aufmerksamkeit der versteckten Symmetrie $G_{2(+2)}/SO(4)$, die bei der dimensionalen Reduktion der $d = 5, \mathcal{N} = 2$-Supergravitation erscheint, und ein Parallelmodell zum bekannteren $E_{8(+8)}/SO(16)$-Fall in der elfdimensionalen Supergravitation darstellt. Aufbauend auf Teilresultaten für den bosonischen Sektor stelle ich eine Ableitung vor, die bosonische wie fermionische Terme einschließt. Dabei klärt sich auf, wie sich aus der dimensionalen Reduktion die versteckte lokale Symmetrie $SO(4)$ ergibt, inklusive einer ungewöhnlichen Eigenschaft, die aus einer Analyse der R-Symmetrie für $\mathcal{N} = 4$-Supersymmetrie und der Supersymmetrievariationen folgt und dieses Modell von seinem elfdimensionalen Verwandten unterscheidet: Das Erscheinen einer zusätzlichen $SO(3)$, die Teil der lokalen Symmetrie, aber nicht mit der dimensionalen Reduktion des Gravitinos assoziiert ist, hat zur Folge, daß direkt bei der dimensionalen Reduktion zunächst nur eine diagonale Untergruppe der Coset-$SO(4)$ sichtbar wird.

Contents

Chapter 1

Introduction

At the present time, supergravity can look back on a long and colourful history. From its inception, and practically by definition, it has been intimately connected with the quest for a quantum theory of gravity. Supersymmetry postulates that the anticommutator of two supersymmetries, Q, generates a translation $\{Q, Q\} \sim P$. In a theory with local supersymmetry, a location-dependent P becomes the generator of diffeomorphisms, a clear signal that the resulting *supergravity* will be connected to Einsteinian gravity. On the other hand, the ability of supersymmetry to alleviate divergencies gave rise to a different hope: Could supersymmetry possibly cure the ills that had doomed to failure the straightforward, "particle physicist" style quantization of ordinary gravity?

While it emerged that matters are not that simple, further developments opened up ever more interesting vistas. Soon, supergravity theories in diverse dimensions were developed that incorporated vector fields, the type of field associated with the non-gravitational forces. While in this way the field of supergravities widened, ground-breaking work on the classification of supersymmetries showed that there was a limit to the diversity. In particular, there exists a maximal supergravity, beyond which no further supersymmetries or extra dimensions can be added – the uniquely defined $\mathcal{N} = 1$ supergravity in eleven dimensions, the pinnacle of a tower of supergravities in various dimensionalities below eleven[1]. Were the supergravities living in higher dimensions than our four-dimensional space-time just abstract mathematical models? The possibility of compactifying higher-dimensional spaces in order to obtain lower-dimensional ones meant that they might be more, and that one might even be able to put the extra dimensions to good use: Compactified dimensions enlarge the particle spectrum of the theory, introducing new internal symmetries. Perhaps the four-dimensional world we see around us could be derived as a compactified descendant of higher-dimensional space-time, with the standard model of elementary particle physics, its various gauge symmetries and rich content of matter particles, the consequence of a higher-dimensional, and preferably simpler, theory? A natural candidate for the higher-dimensional theory was, of course, the maximally supersymmetric model – the

[1] For the analysis, see [101]; for the construction of the eleven-dimensional theory, [35].

eleven-dimensional supergravity.

However, even while the efforts to distill four-dimensional physics from higher-dimensional supergravity were under way, competition emerged in the form of superstring theory. This theory replaced the traditional starting point for formulating quantum theories, namely classical point particles, with one-dimensional strings living in ten dimensions and promised, as a result, a (perturbative) theory that incorporated gravity and that could be quantized without undesirable divergencies. String theory also makes use of extra dimensions, and aims at unifying all known interactions and particles. Following the so-called “first superstring revolution” (kicked off by the proof that superstring theories are, indeed, anomaly free), strings began to supplant maximal supergravity as the community’s favourite candidates for a “theory of everything” – especially as it became clear that four-dimensional physics was not so easily recovered from higher-dimensional supergravity, after all.

Supergravities still had their role to play. Ten-dimensional supergravities could be seen to be the low-energy limits of the different types of string theory. Correspondingly, lower-dimensional supergravity theories were the low-energy limits of compactified string theories. One theory, however, was notably absent. Eleven-dimensional supergravity, of all models, appeared to have no place in the stringy scheme of things.

This changed with the “second superstring revolution”, in 1995. Eleven-dimensional supergravity returned to play a leading role in the context of the conjectured “M-Theory”[2], and renewed interest was sparked in both eleven-dimensional supergravity in particular and in its lower-dimensional kin in general. This is the context in which the present thesis makes its contribution.

In order to complete the backdrop, I will characterize briefly the revival of general interest in eleven-dimensional supergravity in the context of M-theory then go into slightly more detail to provide an introduction to the so-called hidden symmetries of (super-)gravity that will play a central role in the following pages, review some aspects of five-dimensional supergravity and close with a brief outline of the content of the subsequent chapters.

M-theory and the supergravity renaissance

Before 1995, string theory suffered from an embarrassment of riches.[3] Far from there being a unique string theory, there were in fact five seemingly disparate types of ten-dimensional, perturbatively formulated superstring theories: Type I, Type IIA, Type IIB, $SO(32)$-heterotic and $E_8 \times E_8$-heterotic. This picture changed completely in the

[2]The reader interested in retracing in more detail the background of these introductory remarks might want to look at the following literature. The history of the various supergravity theories, from the inception of supergravity up to the late 1980s, can be gleaned from the collection of articles in [122]; for a modern perspective, see [38, 134, 130].

[3]The story of string theory is laid out in the relevant text books, for example the classical tomes by Green, Schwarz and Witten [60, 61] for pre-second-revolutionary string theory, and by Polchinski [114, 115] for a glimpse of more recent developments.

course of what is now called the second superstring revolution, when numerous duality relations between these five string theories were discovered.

Duality relations between different theoretical models are a time-honoured concept.[4] A classical example is the duality between the (quantum) sine-Gordon and the massive Thirring model [25]: The theory of a bosonic field φ with potential $\sin(\beta\varphi)+\gamma$ results, for certain choices of the constants β, γ, in exactly the same physics as the theory of a single, massive Dirac field with a certain quartic potential! In the context of gauge theory, there is the example of Montonen-Olive duality [98], a generalization of the duality between ordinary electric and magnetic fields in a vacuum: In a certain gauge theory, the strongly coupled domain is dual to the weakly coupled domain of that same theory, with electric and magnetic fields, as well as charged particles and certain solitonic "compound particles" exchanging their roles.

The examples illustrate the key feature of duality: Two models that are dual might seem different, but are in fact complementary descriptions of the same underlying physics. Such dualities began to emerge between the different ten-dimensional string theories, from the mid-1980s on[5]: The earliest known duality, "Target space" or, shorter, "T-duality", relates certain seemingly different compactifications of string theories. A simple example is IIA string theory compactified on a circle with radius R, which is dual to the IIB theory compactified on a circle with radius $R' \sim 1/R$, where one theory's momentum modes (modes due to the quantization of the momentum along the circle) are dual to the other's winding modes (modes corresponding to strings wound non-contractably around the circle). Another type of duality, "S-duality", is akin to Montonen-Olive duality (and, in fact, in the modern view, a generalization thereof). It relates the weak-coupling limit of one string theory to the strong-coupling limit either of another, or of the same string theory, and vice versa. While the presence of the strong coupling limit means that the duality might well be impossible to prove – in contrast to T-duality, which can be confirmed in perturbation theory – there are numerous possible checks, notably the matching of spectra using the elementary fields and solitonic objects in the low-energy effective theories. One example for S-dual theories are the $SO(32)$-heterotic and the Type I string theories; an example for a self-S-dual theory is the IIB theory. Just as Montonen-Olive duality does, the string self-S-dualities involve a generalized discrete $SL(2,\mathbb{Z})$ symmetry of the theory, and from an analysis of the interplay of S- and T-dualities arises an even larger discrete structure – "U-duality", a discrete group conjectured to unify S- and T-dualities [69]. Further analysis of this net of dualities suggested an astonishing state of affairs: The strong coupling limits of IIA- and E_8-heterotic theory appear to be linked not to other string theories, but to a theory that lives in one more dimension than the canonical ten for string theories! These observations lead to what is nowadays called the second superstring revolution. At its end stood the vision of all five string theories unified in

[4]The reader interested in a pedagogical introduction to dualities in general is referred to the Trieste lectures by Harvey, [65].

[5]Reviews can be found in [131, 111, 126, 46, 53, 132, 141] and, particularly useful as it includes details on the supergravity aspects I will discuss in more detail below, [39].

an eleven-dimensional “M-theory”, with both M-theory and the string theories richly populated with extended objects (“branes”). While many of the properties of the conjectured M-theory were unclear – and many still are – its low energy limit brought a reunion with an old acquaintance. The eleven-dimensional supergravity was back. Research on various aspects of M-theory has continued at pace ever since, and much of it involves supergravity methods. Although given the conjectured nature of M-theory the focus is on the eleven-dimensional theory, we will see below that lower-dimensional supergravities have a role to play as well. Before, though, let us take a brief look at one of the more fascinating aspects of supergravity theories.

Hidden symmetries

Compactification, as a way to obtain lower-dimensional from higher-dimensional theories, has already been mentioned. Generically, compactifications give rise to new massive states of the theory, called “Kaluza-Klein-states”. This is easily seen in a reduction on a circle – the compactness of the circle dimension suggests a Fourier expansion of the fields with respect to the angle; from the higher-dimensional kinetic term, derivatives acting on the angle produce mass terms inversely proportional to the circle radius. As the extension of the compact space shrinks, the masses increase; shrinking to zero, the masses “go to infinity”. In the interesting cases, there is a sector of the theory that has not gained Kaluza-Klein masses, and from this sector, the massive Kaluza-Klein-sector of the theory now completely decouples. What is left is called a *dimensionally reduced* model. From the way we have introduced dimensional reduction – a way from higher-than-four dimensions to regular space-time – one might think that this procedure is of interest only to aficionados of extra dimensions. However, that is not the case. Formally, the steps taken to examine four-dimensional space-times that exhibit certain symmetries are equivalent to a dimensional reduction. Implementing, for example, axial symmetry by choosing suitable coordinates and then dropping the dependence of the metric on the appropriate angular coordinate is, in effect, a dimensional reduction from four to three dimensions. It was in this context that the coset symmetries associated with dimensional reduction were first discovered [51, 103, 57].

As a simple example, consider the dimensional reduction of pure gravity on an n-dimensional hypertorus (some technical details of which are laid out in chapter 3 of this thesis). The hypertorus is the simplest possible compactification space, and here, the procedure is especially straightforward – dimensional reduction amounts to taking the higher-dimensional theory’s Lagrangian and dropping the dependence on the coordinates associated with the reduced dimensions. From the point of view of the lower-dimensional gravity theory, the metric coefficients associated with the n internal dimensions (those dimensions that have been “compactified away”) are scalars. However, it follows from the symmetries the scalars have inherited from the higher-dimensional theory that they can be written in a way that is, mathematically, very special: The scalar Lagrangian can be rewritten as that of a “coset model” that binds the different scalars into a unified, group theoretical structure. With some technical finesse (namely a separate treatment of the metric determinant, and the introduction of orthonormal vielbeins), it can be shown that the scalar degrees of freedom are governed by a coset structure $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ – a type of space defined by the special linear group associated with the n internal dimensions, in which no distinction is made between elements that are related to each other by orthogonal transformations[6]. Much more surprising is the fact that, in certain cases, an even larger coset structure can be made to appear.

A simple example of this is the reduction of pure gravity from four to three dimensions. The original metric field g_{MN} has two physical degrees of freedom (corresponding to the two polarizations of the four-dimensional graviton). Under dimensional reduction, higher-dimensional space-time indices ($M, N, \ldots$) can be decomposed into lower-dimensional space-time indices ($\mu, \nu, \ldots$) and internal indices $m, n \ldots$. In our special case, the reduction by one dimension, internal indices take on a single value only, which we will call 4. The original metric g_{MN} splits into the new, lower-dimensional space-time metric $g_{\mu\nu}$, a lower-dimensional space-time vector $g_{\mu 4}$, the "Kaluza-Klein-vector", and one lower-dimensional scalar g_{44}. The Kaluza-Klein vector and scalar each account for one physical degree of freedom; the metric $g_{\mu\nu}$ has none, corresponding to the fact that, in three dimensions, gravity has no propagating degrees of freedom. However, there is a twist as, in a three-dimensional space-time, vectors can be Hodge-dualized to scalars: Quite generally, one can replace a p-form field living in d dimensions by a $(d-p-2)$-form field, with the Bianchi identity of the old field leading to the field equations of the new[7]. The dualized theory is, at least on a classical level, equivalent to the original. In the example at hand, the Kaluza-Klein-vector can be dualized in this way to a second scalar B. The crucial point is that the two scalars B and g_{44} have very special properties – it turns out that they parametrize a larger coset structure, namely a $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. The symmetries that appear in the definition of that coset are called *hidden symmetries* of gravity – hidden because they are certainly not evident in the description of the higher dimensional theory: starting from the dimensionally reduced Lagrangian, it takes some reformulation and refinement of the theory to make them manifest.

Symmetry structures such as this have a number of interesting applications to gravitational physics, from solution-generating methods to uniqueness theorems – especially the exceptional (and completely integrable) case $d = 2$, in which the symmetry can be enlarged to reveal a much more complex, infinite-dimensional hidden symmetry structure, the "Geroch group"[8].

The situation is even more interesting if one passes from gravity to supergravity. Again, I will focus on dimensional reduction via hypertorus compactification,

[6]The reader unfamiliar with coset models can find a brief review in section 2.1. The $SL(n,\mathbb{R})/SO(n,\mathbb{R})$ structure is best seen by introducing a vielbein, cf. the introductory section 3 b) of the review [29].

[7]The reader unfamiliar with the procedure can find an example in this thesis on p. 68.

[8]See for example [57, 58, 13, 12], as well as the reviews [92, 109, 105, 68].

which has the great advantage of breaking none of the supersymmetries of the theory. Generically, in supergravity the metric field is joined by one or more additional p-form fields. The dimensional reduction leads to a number of different fields; the higher-dimensional metric contributes scalars and Kaluza-Klein-vectors, and the p-form fields contribute both scalars and form fields of varying degree. Depending on dimension and form degrees, some of the form fields can, again, be dualized to scalars. Naively, one might have expected that, with luck, the additional scalars arising from the p-forms possess an interesting coset structure of their own. However, in a much more wondrous turn of events, it transpires that all the seemingly different scalars arising from the metric and p-forms combine to parametrize *one single coset*, determined by one global and one local hidden symmetry of the supergravity in question[9].

As an example, consider the dimensional reduction of the eleven-dimensional supergravity to three dimensions, summarized in the table below.

Graviton g_{MN} (44)			Photon A_{MNP} (84)				Gravitino Ψ_M (128)	
$g_{\mu\nu}$ (0)	$g_{\mu n}$ (8)	g_{mn} (36)	$A_{\mu\nu\rho}$ (0)	$A_{\mu\nu r}$ (0)	$A_{\mu n r}$ (28)	A_{mnr} (56)	Ψ_μ (0)	Ψ_m (128)

In the top row are listed the fields of eleven-dimensional supergravity and, in parentheses, the associated physical degrees of freedom. We have already mentioned that, under dimensional reduction, higher-dimensional space-time indices $(M, N, \ldots)$ can be decomposed into lower-dimensional indices $(\mu, \nu, \ldots)$ and internal indices $(m, n, \ldots)$ – those indices associated with the new space-time, and extra indices associated with the dimensions of the internal space. For example, the original metric g_{MN} splits into the new, lower-dimensional space-time metric $g_{\mu\nu}$, a lower-dimensional space-time vector $g_{\mu n}$ and lower-dimensional scalars g_{mn}; the original gravitino into the lower-dimensional gravitino and spin-1/2 fermions. On the bosonic side, reduction results in 92 scalar and 36 vector degrees of freedom. Next, one can employ Hodge dualization to obtain as many scalars as possible – both the Kaluza-Klein-vectors $B_\mu{}^m$ and the vectors $A_{\mu np}$ can be dualized to scalars. Afterwards, one is left with 128 scalar fields, which turn out to fit into the coset $E_{8(+8)}/SO(16)$ (the dimension of the E_8 is 248, that of $SO(16)$, 120, the difference indicating how many scalars it takes to parametrize the coset which is, indeed, 128). On the fermionic side, the local enhanced symmetry makes an appearance as well: An analysis of the fermions shows that the internal Spin(8) symmetry associated with the spinor side of the internal space can be enlarged to a local $SO(16)$, with the gravitino transforming in the vector representation **16** and the spin-1/2 fermions in the $\overline{\mathbf{128}}$, one of the spinor representations of the $SO(16)$.

The appearance of the exceptional Lie group $E_{8(+8)}$ is no isolated case, but in fact a generic feature of the dimensional reduction of eleven-dimensional supergravity – by a long shot, the best-studied supergravity, as far as hidden symmetries are concerned. Successive dimensional reduction of this theory [31, 30, 28, 71] reveals a remarkable pattern. As the scalar fields of the resulting supergravity (in each case, that dimension's

[9] For reviews on this topic, see [74, 39, 29].

unique maximal supergravity [123, 36, 101]) assemble themselves into the respective cosets, the global groups are given by the exceptional Lie groups of the E-series and lower-dimensional groups of a similar type[10].

dimension	8	7	6	5	4	3
group	$E_{3(+3)}$ $\simeq SL(3)\times SL(2)$	$E_{4(+4)}$ $\simeq SL(5)$	$E_{5(+5)}$ $\simeq SO(5,5)$	$E_{6(+6)}$	$E_{7(+7)}$	$E_{8(+8)}$.

In the bosonic sector, the reduction steps leading to the exceptional series can be performed in a very systematic and clear way, matching up sets of scalars with the root vectors used in the mathematical analysis of Lie algebras, and showing how the Dynkin diagrams defining those algebras grow with each reduction step[11]. Below three dimensions, matters are again much more complicated. A reduction to two dimensions leads to a generalization of the infinite-dimensional Geroch group, involving cosets of affine Kac-Moody groups [71, 72, 73, 11, 13, 105, 84]. The reduction to one dimension is much less well-understood, although there are proposals for the infinite-dimensional structures to be expected in that case [106, 95].

We have already encountered the connection between eleven-dimensional supergravity and the conjectured M-theory. The proposed correspondence extends to the hidden-symmetries, as well (and, in effect, this part of the correspondence was of great importance in the derivation). The conjecture is that the U-duality groups encountered in the toroidal compactification of the Type II superstrings (one of which is itself a toroidally compactified descendant of M-theory) are discretized versions of their hidden symmetry counterparts, faithfully following the same path down the E-series [69].

Research involving or motivated by the hidden symmetries of supergravity (in most cases the eleven-dimensional supergravity) is currently quite an active field, with results ranging from the gauging of subgroups of the global exceptional groups [110, 41, 5] to the identification of new vacua [76, 52] to possibilities for enlarging the hidden symmetries even further [86]. One line of research that might lead to an especially large dividend involves the lifting of symmetries of the dimensionally reduced eleven-dimensional supergravity back to the original theory. In the dimensional reduction schemes, the dependence of the fields on the internal space coordinates is dropped completely. However, it was realized some time ago that even a dependence on those internal coordinates does not contradict the existence of the enhanced local symmetries that occur in the coset construction. Guided by the construction of the hidden symmetry in the reduction to some lower-dimensional space-time, it is possible to reformulate the eleven-dimensional theory in such a way that it exhibits the enhanced local symmetry in question. The first example for this kind of reformulation took its cue from four dimensions, replacing the tangent space symmetry $SO(10,1)$ in eleven dimensions by

[10] In the table, only $E_{6(+6)}$, $E_{7(+7)}$, $E_{8(+8)}$ are exceptional, i.e. not part of any of the regular types of classical groups. If one continues the pattern that relates their defining Dynkin diagrams, one arrives at groups isomorphic to ordinary, classical groups. Readers interested in the details should consult the introductory literature on Lie algebras, e.g. [70, 55].

[11] For the eleven-dimensional case, see [32, 33]; for the general reduction to three dimensions, [34].

a local symmetry $SO(3,1)\times SU(8)$ [40]. Guided by the reduction to three dimensions, there exists a reformulation with tangent space symmetry $SO(2,1) \times SO(16)$ [104], and analogous constructions have been performed for the reduction to five or six dimensions [48]. These reformulations are achieved by the introduction of "generalized vielbeins" which inherit properties from the global hidden symmetry. For instance, coming from three dimensions (with the coset $E_{8(+8)}/SO(16)$), generalized vielbeins can be defined that contain parts of the original vielbein, but also of the tensor degrees of freedom, and whose properties hint at an "exceptional geometry" present even in the full eleven-dimensional theory [85, 83]. This is of special interest because of the possible consequences for the search for the elusive M-theory. Could this reformulation, shifting the physical degrees of freedom from a usual vielbein and p-forms to a vielbein governed by exceptional geometry help, either in one of the finite-dimensional incarnations, or even in the form of an infinite-dimensional generalization [107, 108, 138, 66]? Even with the jury still out on these far-reaching potential consequences, the hints of an "exceptional geometry" are intriguing enough.

Supergravities in five dimensions

We have already encountered in passing a five-dimensional supergravity with exceptional symmetry E_6, reduced from the eleven-dimensional theory [35]. This is, in fact, the way that the first five-dimensional supergravity, the maximal five-dimensional supergravity with $\mathcal{N}=8$ was formulated in [28]. Further research has widened the field, with some of the five-dimensional supergravities obtainable from the maximal model by truncation, others not [62, 63].

There are several reasons why five-dimensional supergravities have flourished in the M-theory inspired supergravity renaissance. One is the fact that compactification of eleven-dimensional M-theory to five dimensions is the analogue of the well-studied compactification of string theory to four dimensions, involving the same Calabi-Yau manifolds that feature so prominently in string theory [142]. Another context in which a five-dimensional (AdS-)supergravity plays a leading role is the famous AdS/CFT correspondence, the conjectured duality of said supergravity (times a spherically compactified space) and a conformal field theory living on the AdS boundary [47, 81]. Also, in the context of large extra dimensions ("brane worlds"), a concept from the 1980s that has experienced a renaissance of its own in the context of string-/M-theory, the simplest example is a regular four-dimensional space-time plus one extra dimension [3, 121, 118, 119].

In addition, there is a much more down-to-earth reason for those interested in eleven-dimensional supergravity to be mindful of its five-dimensional kin. There is a five-dimensional theory that is in many respects the "little brother" of the eleven-dimensional version, sharing many of its properties [28, 18, 96]. It is the (ungauged) $\mathcal{N}=2$ supergravity, which can be obtained from the $\mathcal{N}=8$ theory by consistent truncation[12]. Like the eleven-dimensional theory, the $d = 5, \mathcal{N} = 2$ supergravity contains only the graviton, gravitino (albeit with an extra index) and one type of p-form field. Where the eleven-dimensional theory has a four-form field, the five-dimensional theory sports a more modest two-form, but apart from that, the Lagrangian structure is the same, with both theories containing a Chern-Simons term and characteristic couplings of gravitino to p-form, in addition to the usual kinetic terms. The $d = 5, \mathcal{N} = 2$ is the simplest possible supergravity in five dimensions – there is no $\mathcal{N} = 1$ theory[13] – just as the eleven-dimensional theory is the minimal supergravity with its dimensionality (although the parallel is incomplete, as the eleven-dimensional supergravity is also the maximal possible theory, whereas higher extended supersymmetries are possible in five dimensions). These similarities have led to a number of attempts to learn more about one theory by studying the other, from toy models of the M5 brane [9] and cosmological models [20] to the (albeit only partially successful) attempt to use the theory as a simplified proving ground for methods used to study the U-dualities of M-theory [97, 125]. In view of what was said above about the rich hidden-symmetry structure of the eleven-dimensional theory, the obvious question is: What about the hidden symmetries of $d = 5, \mathcal{N} = 2$ supergravity?

Hidden symmetries in d=5, N=2 supergravity, including an outline of the thesis

There are indeed results concerning the hidden symmetries of $d = 5, \mathcal{N} = 2$ supergravity. An early work of Chamseddine and Nicolai has studied the dimensional reduction to four dimensions, obtaining $d = 4, \mathcal{N} = 2$ supergravity coupled to an $SO(2)$ vector matter multiplet [18] – a toy model for the $d = 4, \mathcal{N} = 8$ supergravity [31]. More interesting is the reduction from five to three dimensions – the analogue of the reduction that led to the discovery of the exceptional $E_{8(+8)}/SO(16)$ structure of the eleven-dimensional theory. Quite generally, in $\mathcal{N} = 2$ supergravity, the presence of supersymmetry poses severe restrictions for the possible coset models (or their generalization, nonlinear sigma-models). These have been utilized in a far-reaching analysis, yielding classifications of the pertinent nonlinear sigma models in terms of manifolds with "special geometry" [45, 44]. For the special case $d = 5 \to d = 3$ that I am interested in, it turns out that the coset is $G_{2(+2)}/SO(4)$ – just as in the eleven-dimensional case, an exceptional over an orthogonal group. More concretely, Mizoguchi and Ohta [96] have studied the bosonic part of the model using a decomposition of $G_{2(+2)}$ with respect to $SL(3, \mathbb{R})$, in which the $SO(4)$ can be singled out by its symmetry under a certain involution. With an ansatz for fitting the scalar fields of the reduced supergravity into $SL(3, \mathbb{R})$-representations, they were able to express

[12]This was the way the theory was, in fact, first obtained in [28]; almost in parallel, it was formulated directly in [18].

[13]Strictly speaking, this is convention-dependent – in some conventions, the minimal supersymmetry is always called the $\mathcal{N} = 1$. Our convention counts the values of the non-spinorial index of the gravitino.

the bosonic Lagrangian in terms of the coset parameters (this is one ingredient of the work on discretized U-dualities [97, 125] already cited). Last but not least, with their systematic construction briefly mentioned above, Cremmer, Julia, Lü and Pope have shown [34] how the root vectors constructed from the dimensionally reduced scalars in five dimensions form the G_2 Dynkin diagram.

As valuable as those results are, they leave a number of open questions. Notably absent in those studies are the fermions. How does the enhanced local symmetry arise from the dimensional reduction of the spinors? How is it implemented for the matter fermions? What are the redefined supersymmetry variations? What is the explicit formulation of the coset kinetic terms in terms of the local $SO(4)$ symmetry? What are the gauge covariant connections? Answers to these questions are interesting beyond a better understanding of the dimensionally reduced theory. I have discussed above the possible lifting of symmetries to higher dimensions. The constructions used for recovering the generalized vielbeins make ample use of the fermionic side of the theory, such as the reformulated gamma matrices and the supersymmetry variations. Thus, the fermionic terms are a prerequisite for possible studies of analogues of the exceptional geometries in five dimensions.

This is the motivation behind the main aim of this thesis: To analyse the dimensional reduction of $d = 5, \mathcal{N} = 2$ supergravity, both fermions and bosons, identifying the coset model in $d = 3$ dimensions, including the kinetic term and the compound connection coefficients, and to demonstrate the emergence of the hidden symmetry in the fermionic sector. There are, in principle, two ways to achieve this. One could attempt to shift the consistent truncation by which the $d = 5, \mathcal{N} = 2$ arises from the maximal $d = 5, \mathcal{N} = 8$ theory to the three-dimensional case and to a direct truncation of the $E_{8(+8)}/SO(16)$ to the $G_{2(+2)}/SO(4)$ model. The second way, which is the one chosen in this thesis, is akin to the way the hidden symmetries in the $d = 11 \to d = 3$ theory were found in the first place: Starting from the $d = 5, \mathcal{N} = 2$ theory, and explicitly identifying local and global hidden symmetries in the reduction to three dimensions. The plan of action is as follows.

In chapter 2, I briefly review pertinent concepts from the theory of coset models and then proceed to construct the three-dimensional $G_{2(+2)}/SO(4)$ coset supergravity. This model later serves as a guidepost for the subsequent dimensional reduction, introducing the scalar objects P_μ and connection coefficients Q_μ that the dimensionally reduced scalars need to fit into. The construction uses a coset decomposition of $G_{2(+2)}$ that is derived in appendix B. As the comparison with the reduced model does not depend on higher fermionic terms, quartic and higher terms in the Lagrangian are neglected throughout.

In chapter 3, I review the basic concepts and formalism of the dimensional reduction of supergravity theories. The analysis is kept more general than strictly necessary, both in order to show which of the shared similarities of the E_8- and G_2-constructions arise from the common target dimensionality alone, and in the hope that the general expressions derived, e.g. for the reduction of Rarita-Schwinger terms, might save some work in the future construction of similar models.

In chapter 4, the results of the preceding chapters are put to work: After a review of the $d = 5$, $\mathcal{N} = 2$ theory, I proceed with the dimensional reduction. From an analysis of the gamma matrices, it is shown how the spinorial quantities fit into the proper group representations of the reduced theory. Then starts the matching of the reduced theory with the model derived in chapter 2, first fixing certain parameters that had been left free in both theories and identifying the compound fields P_μ and Q_μ. With the correspondence fixed, it is time for several cross-checks. This concludes the main part of the thesis.

Chapter 5 is a mathematical afterthought, a spin-off from the analysis of super-charges contained in chapter 2. Such analyses hint at quaternionic spinor structures, and in this chapter I take a look at possibilities to make these visible in the fermionic parts of field theoretical formulations.

Chapter 6 contains a summary of the results and discusses some further directions for research.

Chapter 2

Constructing the $G_{2(+2)}/SO(4)$-supergravity in three dimensions

In this section, the $G_{2(+2)}/SO(4)$-supergravity is constructed, which will later serve as a point of reference for the dimensional reduction of the five-dimensional theory. Quartic and higher fermionic terms in the Lagrangian will be ignored throughout. The plan of action is dictated by the model under construction: Choose the fields, identify their representations with respect to the groups involved, make an ansatz for the Lagrangian and the supersymmetry variations, and fix the details by demanding local supersymmetry.

In light of this, the chapter is structured as follows: Section 2.1 is a brief and somewhat selective review of the basic ingredients and fundamental definitions common to coset models, focusing on results to be used later on. Section 2.2 introduces the basic fields and, aided by an analysis of the supercharges and supersymmetry variations, derives to which group representations the fields belong. The results are then implemented in section 2.3, in which these representations are translated into a proper index structure (using the representations constructed in appendix B), including the definition of covariant derivatives and identities for the symplectic spinors used. With this input, the construction of the theory's Lagrangian and supersymmetry variation takes place in section 2.4, while the results are summarized for later reference in section 2.5.

2.1 Coset models – a brief review

Coset models, a special subclass of nonlinear σ-models, occupy the middle ground between linear models, easy to handle but a far cry from the complex, nonlinear dynamics of the real world, and more general non-linear models, more realistic but, in consequence, well-nigh intractable. Coset models incorporate nonlinearity, yet in a way that is thoroughly grounded in group theory, making the resulting models somewhat tractable. The following will be a brief and selective introduction to certain properties of such models, with the focus on what will be needed for the subsequent computations: After a few remarks on the relevant group theoretical concepts, I will directly introduce the relevant objects and their relations in the classical, unitary-gauge formulation [26, 15, 93] that will be used in the construction of the specific coset model in the later sections.

There are several ways to build interesting field theories involving scalar fields. Notably, one can assume that those scalars parametrize some (pseudo-)Riemannian manifold – an assumption that leads to a number of interesting consequences, as the field theory inherits certain geometrical properties from the scalar manifold. As a special case, one can assume that the scalar manifold in question is a *homogeneous space*, in other words: there is some continuous group G whose action is an isometry on the manifold, and the action of the group is *transitive* – for every two points in the manifold, there exists a group element taking one into the other. With this transitivity, it is possible to define the space exclusively in terms of the group: From some starting point x_0 on the manifold, every group element can be mapped to the point $x = gx_0$ reached by the corresponding action on the manifold. However, the correspondence is not one-to-one: Denote by H the subgroup of G that leaves x_0 invariant. Then a group element g and a group element gh, with $h \in H$, map x_0 to the same point $x = gx_0$. The point x does not correspond to a unique group element g, but to a whole (left) *coset* gH of G; the equivalence class of group elements that are mapped to each other by right multiplication with some $h \in H$. In turn, the homogeneous space corresponds to the *coset space* formed by all such cosets in G. It is such a coset space that we shall set out to describe in the following.

Let us start with a non-compact real form of a d-dimensional Lie group G, as well as a d_h-dimensional maximal compact subgroup H of G. While the following arguments can, with some extra effort, cast into an abstract form valid for general Lie groups and algebras, let us, for conceptual simplicity, assume that we are already dealing with matrix representations of G and H. Let X_i, with $i = 1, \ldots, d_h$, be the generators of $\mathfrak{h}$, the Lie algebra of H, and let Y_m, $m = 1, \ldots, d - d_h$ be a set of generators that complement the X_i to form the full Lie algebra $\mathfrak{g}$ of G. Commuting two compact generators or commuting two non-compact generators gives a linear combination of compact generators, while the commutator of a compact with a non-compact generator is a linear combination of non-compact generators; written in terms of structure constants,

$$
\begin{aligned}
[X_i, X_j] &= f_{ij}{}^k X_k, \\
[X_i, Y_m] &= f_{im}{}^n Y_n, \\
[Y_m, Y_n] &= f_{mn}{}^i X_i,
\end{aligned}
\tag{2.1}
$$

where all other structure constants are zero. The first commutator property is just the subalgebra property; that the commutator of a compact and a non-compact generator be a linear combination of non-compact generators can always be achieved with a suitable choice of basis; that the commutator of two non-compact generators is a linear combination of compact generators makes G/H a *symmetric* space, and is always true for G simple, and H a maximal subgroup [59, Ch. 9,IV].

According to a basic result in the theory of Lie groups, it is always possible to find a neighbourhood of the identity of G where every group element can be written with the help of the exponential map as

$$g = e^{\varphi\cdot Y + a\cdot X} = e^{\hat{\varphi}\cdot Y} e^{\hat{a}\cdot X}, \tag{2.2}$$

where $a \cdot X \equiv a^i X_i$ etc. are abbreviations for linear combinations of generators and coefficients (a notation frequently used in the following)[1]. If a and φ (or, in the equivalent notation, $\hat{a}$ and $\hat{\varphi}$) are constant, g is a group element. If we choose them to be space-time-dependent fields, g defines a group-valued field. In order to define the coset model, we will look at fields whose physical degrees of freedom parametrize that particular neighbourhood of G/H. This allows the compact part of G, namely H, to be retained as a gauge symmetry, while the non-compact part of G acts like a global symmetry.

One possibility of realizing such fields is to take the above expression for g and simply set $a^i = 0$, sometimes called "unitary gauge". While this simplifies the parametrization, the way that elements of G act on parameter fields becomes somewhat more complicated. As (2.2) defines a group element, it is natural to have elements of G act on g by left multiplication. The result of such action will also be of the form $e^{\varphi'\cdot Y} e^{a'\cdot X}$, however, a' will not, in general, be equal to zero. Still, given any $g' = e^{\varphi'\cdot Y} e^{a'\cdot X}$, one can always find a group element $h' \in H$ so that $g'h' = e^{\varphi'\cdot Y}$ is in the proper gauge, namely $h' = e^{-a'\cdot X}$. Hence, it makes sense to modify the transformation of a group element $\bar{g}$ acting from the left on g by including with each such action a $\bar{g}$-dependent transformation $e^{\Sigma(\bar{g})} \in H$ that acts from the right and ensures that g retains its proper form. The explicit form of $\Sigma(\bar{g})$ can be calculated by looking not at a full transformation with $\bar{g} = \exp(\alpha \cdot Y + \beta \cdot X)$, but at a corresponding infinitesimal transformation where only terms linear in the coefficients α_i, β_i are kept. The result is[2]

$$\Sigma \;\; = \;\; \tanh(\mathrm{ad}_{\varphi}/2)(\alpha \cdot Y) - (\beta \cdot X), \tag{2.3}$$

where the right hand side is defined as a power series in the adjoint map $\mathrm{ad}_{\varphi} : X \mapsto [\varphi, X]$. The highly nonlinear action of the transformation on the generating field φ turns out to be

$$\delta\varphi \;\; = \;\; \frac{\mathrm{ad}_{\varphi}}{\tanh(\mathrm{ad}_{\varphi})}(\alpha \cdot Y) - \mathrm{ad}_{\varphi}(\beta \cdot X). \tag{2.4}$$

In the definition of Σ, it is notable that the compensating transformations are explicitly field dependent.

In order to use the field e^{φ} to define a field theory, it will be necessary to define a kinetic term governing its behaviour. In preparation, let us take a look at the derivative

[1]A proof for this can be found, for example, in chapter VI of [24].

[2]The following equations are derived in appendix A.1, p. 85f.

expression $e^{-\varphi}\partial_\mu e^{\varphi}$, which decomposes into a compact part Q_μ and a non-compact part P_μ. They can be explicitly written down as[3]

$$P_\mu = \frac{\sinh[\mathrm{ad}_\varphi]}{\mathrm{ad}_\varphi}\,\partial_\mu\varphi,$$
$$Q_\mu = \frac{1-\cosh[\mathrm{ad}_\varphi]}{\mathrm{ad}_\varphi}\,\partial_\mu\varphi. \qquad (2.5)$$

The transformation behaviour of the expression $e^{-\varphi}\partial_\mu e^{\varphi}$ can be examined using the same infinitesimal version of the transformation induced by $e^{\alpha\cdot Y+\beta\cdot X}$, above, and it follows that

$$\delta(e^{-\varphi}\partial_\mu e^{\varphi}) = \partial_\mu\Sigma + [e^{-\varphi}\partial_\mu e^{\varphi},\Sigma] \qquad (2.6)$$

which, splitting again into compact and non-compact components, translates to

$$\delta P_\mu = [P_\mu,\Sigma]$$
$$\delta Q_\mu = \partial_\mu\Sigma + [Q_\mu,\Sigma]. \qquad (2.7)$$

In words: P_μ transforms in the adjoint representation of the local H-symmetry, while Q_μ transforms as one would expect of a gauge field. The view of the composite field Q_μ as a gauge field suggests the definition of a gauge covariant derivative D_μ as

$$D_\mu := \partial_\mu + [Q_\mu,\,.\,]. \qquad (2.8)$$

From (2.7), it can also be read off how to construct a simple invariant term that is a good candidate for the kinetic part of a Lagrangian: Given the Killing form κ associated with the Lie algebra $\mathfrak{g}$, the expression $\kappa(P_\mu, P^\mu)$ is invariant under the adjoint action of Lie algebra elements and has all the trimmings of a good, albeit highly non-linear, kinetic term: From the definition of P_μ given in (2.5), it can be seen that, to lowest order in φ, it is simply $\kappa(\partial_\mu\varphi, \partial^\mu\varphi)$, which, up to proper diagonalization to separate the different physical degrees of freedom, is just a standard kinetic term for scalar fields.

Next, a look at the supersymmetry transformations. Denote the supersymmetry variation of the linearized theory by $\delta_{SL}\varphi$. By definition, $\delta_{SL}\varphi$ should be in the non-compact part of the algebra. This suggests the ansatz

$$\frac{1}{2}(e^{-\varphi}\delta_S e^{\varphi} - e^{\varphi}\delta_S e^{-\varphi}) = \delta_{SL}\varphi, \qquad (2.9)$$

which, decomposed into powers of ad_φ, is seen to be non-compact, and which furthermore, when expanded into a power series in φ, reproduces the linear result. Using eq. (A.2) to solve for $\delta\varphi$, we obtain

$$\delta_S\varphi = \frac{\mathrm{ad}_\varphi}{\sinh(\mathrm{ad}_\varphi)}\,\delta_{SL}\varphi. \qquad (2.10)$$

[3] A derivation is found in appendix A.1, p. 85.

This, in turn, can be used to compute the supersymmetry variation of P_μ and Q_μ to be

$$\begin{aligned} \delta_S Q_\mu &= D_\mu \Sigma_S + [P_\mu, \delta_{SL}\varphi], \\ \delta_S P_\mu &= D_\mu \delta_{SL}\varphi + [P_\mu, \Sigma_S], \end{aligned} \tag{2.11}$$

with the definition

$$\Sigma_S := -\tanh(1/2 \cdot \mathrm{ad}_{\,\varphi})\,\varphi. \tag{2.12}$$

In closing, we can derive the following identities: Define an operator $\mathfrak{D}_\mu$ acting on a scalar χ from the left by $\mathfrak{D}_\mu : \chi \mapsto e^{-\varphi}(\partial_\mu e^{\varphi}\chi)$. Direct computation shows that its field strength vanishes, $[\mathfrak{D}_\mu, \mathfrak{D}_\nu]\chi = 0$ for all χ. Writing the operator as $\mathfrak{D}_\mu = \partial_\mu + Q_\mu + P_\mu$ and splitting $[\mathfrak{D}_\mu, \mathfrak{D}_\nu]$ into its compact and non-compact parts, the result is

$$[\partial_\mu + Q_\mu, \partial_\nu + Q_\nu] + [P_\mu, P_\nu] = 0, \tag{2.13}$$

$$(D_\mu P_\nu) - (D_\nu P_\mu) = 0. \tag{2.14}$$

2.2 Basic fields and group theoretical considerations

Having defined the objects to be used in the description of the coset model, let us set about constructing a locally (on-shell) supersymmetric model, living in three space-time dimensions and based on the coset $G_{2(+2)}/SO(4)$. We start with general considerations: What are the basic fields? What are their representations with respect to the symmetry groups involved?

First of all, there will be the scalar fields parametrizing the coset, which will be collectively denoted as φ. They will be present in the Lagrangian in the form of the P_μ, defined above in eq. (2.5), with a kinetic term $\kappa(P_\mu, P^\mu)$. On-shell, each scalar has one physical degree of freedom. Secondly, there is the metric field or, equivalently, a dreibein field $e_\mu{}^\alpha$, represented in the Lagrangian by an Einstein-Hilbert term. As we are in a three-dimensional space-time, the dreibein has no propagating degrees of freedom. Next, to satisfy the requirements of supersymmetry, there must be fermionic partner fields. For the vielbein, there is a gravitino Ψ, with a Rarita-Schwinger term as its kinetic term (and, as is its superpartner, with no propagating degrees of freedom in three dimensions). Finally, we need superpartners for the matter fields φ; they are spin-1/2 fields χ. As complex spinors, each of these spin-1/2 fields would have 2 physical degrees of freedom; as real Majorana spinors, only one.

Next, for supersymmetry and other matters of representations. We are dealing with extended supersymmetry, and our supersymmetry parameters and gravitini carry an extra index to reflect this fact. In general, with this index are associated transformations that commute with the Lorentz group and leave the supersymmetry algebra invariant, forming what is called the *R-symmetry*. In analogy to the $E_{8(+8)}/SO(16)$ case, one might hope to identify the R-symmetry that is associated with the supersymmetry present here with the local symmetry H of the σ-model. This would argue that, in order to obtain the $\mathfrak{so}(4) \simeq \mathfrak{so}(3) + \mathfrak{so}(3)$ that needs to be incorporated in the coset construction, it is necessary to start with $\mathcal{N} = 4$ supersymmetry. While matters will turn out somewhat different, this proves to be the correct ansatz. In order to find out more about the R-symmetry, it is convenient to take a more general look at massless $\mathcal{N} = 4$, $d = 3$ supersymmetry multiplets.

This is a standard technique for exploring the consequences of supersymmetry (and, as such, an integral part of the introductory literature, e.g. chapter 5 in [54] or chapter II in [137]); the analysis we recount here parallels that contained in section 2 of [42]. For definiteness, let us choose the real metric with signature $(-,+,+)$, define three-dimensional gamma matrices in terms of the usual Pauli matrices as $\gamma^0 = -i\sigma_2, \gamma^1 = \sigma_1, \gamma^2 = \sigma_3$, and choose a charge conjugation matrix $C = \sigma_2$ and spinors obeying the Majorana condition $(\overline{\Psi})^T = -C\Psi$. The standard supersymmetry algebra tells us that the anticommutator of two supersymmetry generators (or *supercharges*) Q and $\overline{Q}$ is proportional to the four-momentum operator P, namely

$$\{Q^I_a, \overline{Q}^J_b\} = -2i\delta^{IJ}\,(\gamma^\mu)_{ab}P_\mu \tag{2.15}$$

(with $a, b = 1, 2$ spinor indices, $I, J = 1, \ldots, N$ indices for the R-symmetry, and μ a curved space-time index). In order to study massless multiplets, let us go into an inertial frame in which the energy-momentum of given states is $P^\mu = (E, E, 0)$ for some positive energy E. Using this and the reality condition for the spinors Q, we see that, restricting the operator action to these massless states, we can rewrite the algebra relation (2.15) as

$$\{Q^I_a, Q^I_b\} = 2E\delta^{IJ}\,(\mathbb{1} + \sigma_3)_{ab}. \tag{2.16}$$

In the analysis, it proves useful to look at the physical degrees of freedom as one would in a quantum field theory – as one-particle-states in some appropriate Hilbert space. The results can usually be carried back to the classical description of the physical degrees of freedom.

In order for the Hilbert space of states to be positive definite, as it should be, it is evident that all operators Q^I_2 should annihilate all states. What is left are the real charges Q^I_1, and those have an interesting property: From (2.16), it follows that, up to suitable rescaling, they satisfy the relation $\{Q^I_1, Q^J_1\} = 4E\delta^{IJ}$ of a real Clifford algebra $\mathcal{C}(N, 0)$ with all-positive signature. Let us denote the rescaled operators as Γ^I. There is another operator that we can tie in: with the Fermion number operator F, we can build in the usual way an operator $\boldsymbol{F} = (-)^F$ that has eigenvalues -1 on fermionic and $+1$ on bosonic states. With the supersymmetry generators mapping fermions to bosons, $\boldsymbol{F}$ anticommutes with every one of them; also, per definition, $\boldsymbol{F}^2 = 1$. Thus, we can include it with the generators Q^I_1 to form a larger Clifford algebra $\mathcal{C}(N + 1, 0)$. This is fortunate, as the representation theory of real Clifford algebras is well-known (we will encounter some branches of it later on in chapter 5) and can be used to construct supermultiplets as irreducible representations of the non-zero supersymmetry operators Q^I_1 and of $\boldsymbol{F}$. It is natural to choose a basis in which fermionic and bosonic states are unmixed, in other words: all states are eigenvectors of

$\boldsymbol{F}$, and we can choose, given such a basis with dimension $2n$, $\boldsymbol{F} = \mathbb{1}_n \otimes \sigma_3$. If we write all matrices as block matrices (with $\boldsymbol{F}$ split into one block $\mathbb{1}_n$ and one $-\mathbb{1}_n$), matrices that anticommute with $\boldsymbol{F}$ are block-off-diagonal. For the case of interest here, namely $\mathcal{N} = 4$, one can choose the two possible irreducible representations as

$$\begin{aligned} \boldsymbol{F} &= \sigma_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \\ \Gamma^1 &= \epsilon_c \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \\ \Gamma^2 &= \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \\ \Gamma^3 &= \sigma_1 \otimes \sigma_3 \otimes \mathbb{1}_2 \\ \Gamma^4 &= -\sigma_3 \otimes \sigma_3 \otimes \sigma_2, \end{aligned} \tag{2.17}$$

where $\epsilon_c = \pm 1$ serves to distinguish between the two representations. The structure of this representation is well-known to any student of ordinary quantum field theory – up to signature, the Γ^I are like the ordinary, four-dimensional Dirac gamma matrices, with the $\boldsymbol{F}$ acting like the matrix γ^5, and the choice of $\boldsymbol{F}$ made here corresponds to the usual chiral representation: Bosons and fermions transform as spinors of opposite chirality, with $\mathbb{1} \pm \boldsymbol{F}$ a projector onto the bosonic/fermionic subspaces. The four Γ^I can be used to generate the whole multiplet from any one of its states. The way that single Γ^I act implements supersymmetry on the states thus generated. But also the products $\Gamma^{IJ} = 1/2 \cdot [\Gamma^I, \Gamma^J]$ act on the states, mapping bosons and fermions each to their kin, and thus implementing the R-symmetry on the states, as well. In this respect, the fact that bosons and fermions correspond to spinors of different chirality has an important consequence: As can be read off the representation matrices, the spin group generated by the Γ^{IJ} decomposes into a subgroup generated by $\Gamma^{12} + \Gamma^{34}$, $\Gamma^{23} + \Gamma^{14}$ and $\Gamma^{31} + \Gamma^{24}$ acts on the bosons only, while its counterpart, generated by $\Gamma^{12} - \Gamma^{34}$, $\Gamma^{23} - \Gamma^{14}$ and $\Gamma^{31} - \Gamma^{24}$, acts only on the fermions. The first is the self-dual, the second the anti-self-dual part of the $\mathfrak{so}(4)$, corresponding to the two components in the decomposition $\mathfrak{so}(4) \simeq \mathfrak{so}(3) + \mathfrak{so}(3)$ (where, by common abuse of notation, I take the meaning of $\mathfrak{so}(n)$ to include that of the corresponding spin group, seeing that we're dealing with spinor representations). Let us denote these two $\mathfrak{so}(3)$ by indices indicating whether the group in question acts on bosons, $\mathfrak{so}(3)_B$, or fermions, $\mathfrak{so}(3)_F$.

With this additional information, we can now go back to the question of the transformation behaviour of the fields involved. Guidance is provided by the general rules for the linear part of the supersymmetry transformations: Gravitini transform as some kind of derivative of the supersymmetry parameter ϵ; matter fermions transform as matter bosons times the supersymmetry parameter, and vice versa, schematically:

$$\begin{aligned} \delta_S \Psi &\sim D\epsilon \\ \delta_S \chi &\sim \varphi\epsilon \\ \delta_S \varphi &\sim \bar{\epsilon}\chi. \end{aligned} \tag{2.18}$$

This imposes restrictions on the choices of representation for the three fields φ, χ, Ψ. First of all, the supersymmetry parameter transforms as the operators Q^I do – as a vector of $\mathfrak{so}(4)$, and thus as a spinor both under $\mathfrak{so}(3)_B$ and $\mathfrak{so}(3)_F$. The gravitino, given its supersymmetry transformation behaviour, must transform in the same manner. If we choose the matter fields to lie in the fundamental multiplet we have constructed above using the Clifford representation of the Q^I, that would mean that $\mathfrak{so}(3)_B$ only acts on the bosons φ, and $\mathfrak{so}(3)_F$ on the fermions χ.

Next, we use the fact that we have additional information about the scalars: They are meant to parametrize the coset $\mathfrak{g}_{(2(+2)}/\mathfrak{so}(4)$. From the tensor decomposition of the **14**-representation (explicitly constructed in appendix B), we know how the coset elements are to transform under the $\mathfrak{so}(4) = \mathfrak{so}(3) + \mathfrak{so}(3)$ used in the coset construction: It can be found in the literature [94] or, equivalently, using available Lie group theory software[4], that the group decomposes under the $\mathfrak{so}(4) = \mathfrak{so}(3) + \mathfrak{so}(3)$ as[5] $\mathbf{14} = (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{1}) + (\mathbf{4}, \mathbf{2})$. The generators of the two $\mathfrak{so}(3)$ are in $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{3}, \mathbf{1})$, respectively, so the coset representatives transform in the $(\mathbf{4}, \mathbf{2})$.

This already tells us that we must be dealing with more than two copies of $\mathfrak{so}(3)$: Only one of the two $\mathfrak{so}(3)$ from R-symmetry acts on the bosons; yet the scalars transform non-trivially under both $\mathfrak{so}(3)$ of the local H-symmetry. The pairs of groups cannot be identified with each other, and at least one additional group is needed whose Lie algebra let us denote as $\mathfrak{so}_2(3)$. Still, one can attempt to economize by defining $\mathfrak{so}(3)_B$ to be part of the coset construction, taking the $\mathfrak{so}(4)$ used in that construction to be $\mathfrak{so}(4) = \mathfrak{so}(3)_2 + \mathfrak{so}(3)_B$. Using (2.18), we then arrive at a scheme in which the gravitini Ψ, the scalar fields φ and the matter fermions χ transform under the three $\mathfrak{so}(3)$ as follows:

	$\mathfrak{so}(3)_F$	$\mathfrak{so}(3)_B$	$\mathfrak{so}(3)_2$
Ψ/ϵ	**2**	**2**	**1**
φ	**1**	**2**	**4**
χ	**2**	**1**	**4**

With this information, we can proceed to the construction itself.

2.3 Preliminaries and definitions

In the following, let $\mu, \nu, \rho, \ldots$ be curved indices of three-dimensional space-time, and $\alpha, \beta, \ldots$ flat indices. Let us choose a metric signature $(-,+,+)$, corresponding to the real Clifford algebra $\mathcal{C}(2,1)$, so we can choose the gamma matrices γ^α to be real. The Clifford algebra is semi-simple; let us choose the irreducible representation for which $\gamma^0\gamma^1\gamma^2 = \epsilon_3$ for some $\epsilon_3 = \pm 1$ (for concreteness, a possible choice is $\gamma^0 = \pm i\sigma_2$, $\gamma^1 = \sigma_1$ and $\gamma^2 = \sigma_3$, with σ_i the usual Pauli matrices). We will deal with Grassmann

[4] I have made use of A.M. Cohen, M.A.A. van Leeuwen and B. Lisser's software package *LiE*, available online at [`http://young.sp2mi.univ-poitiers.fr/~marc/LiE/`].

[5] Here and elsewhere, I follow the convention of boldface numbers denoting representations of the corresponding dimension, where in $(\mathbf{m}, \mathbf{n})$, $\mathbf{m}$ denotes a representation of the first, $\mathbf{n}$ of the second $\mathfrak{so}(3)$.

spinors only, and use the convention that complex conjugation reverses the order of Grassmann factors.

In supergravity, one needs to describe spinorial degrees of freedom living in curved space-time, and this is best achieved using a vielbein formalism. For this, one introduces a vielbein field $e_\mu{}^\alpha$ and a flat Minkowski metric $\eta_{\alpha\beta}$ related to the spacetime metric $g_{\mu\nu}$ by $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta}$. On the flat (tangent space) indices, a Lorentz symmetry acts; a covariant derivative can be introduced with the help of the so-called spin connection $\omega_{\mu\alpha\beta}$, and via the Lorentz group and its representations, it is possible to define the necessary spinorial objects needed to describe the fermions of the theory. This is the context for the following definitions of spinors and vielbeins that will be used to describe our three-dimensional model[6].

Next, for the three different $\mathfrak{so}(3)$ introduced in the previous section. I use indices $i, j, \ldots$ as spinor indices of the $\mathfrak{so}(3)_F$, indices $\bar{a}, \bar{b}, \ldots$ for the $\mathfrak{so}(3)_B$ and indices $\dot{a}, \dot{b}, \dot{c} \ldots$ for the remaining $\mathfrak{so}(3)_2$. The representations needed are simple: the **2** is the (fundamental) spinor representation, and the **4** can be written as a tensor representation with three totally symmetric fundamental indices. With this assignment, the scalars have the index structure $\varphi^{\bar{a}\,\dot{a}\dot{b}\dot{c}}$ (symmetric in $\dot{a}\dot{b}\dot{c}$); the matter fermions carry indices $\chi^{i\,\dot{a}\dot{b}\dot{c}}$ (again, symmetric in $\dot{a}\dot{b}\dot{c}$) and the gravitini and supersymmetry parameters are $\Psi^{i\bar{a}}$ and $\epsilon^{i\bar{a}}$, respectively.

The coset construction involves the algebra $\mathfrak{so}(3)_2 + \mathfrak{so}(3)_B$ as the maximal compact subalgebra $\mathfrak{so}(4)$ of $\mathfrak{g}_{2(+2)}$. Decomposed with respect to representations of that subalgebra, an algebra element can be written as a contraction of coefficients with generators E, with the part in $\mathfrak{so}(3)_2$ given as $M^{\dot{a}}{}_{\dot{b}} E^{\dot{b}}{}_{\dot{a}}$, the $\mathfrak{so}(3)_B$ as $N^{\bar{a}}{}_{\bar{b}} E^{\bar{a}}{}_{\bar{b}}$ and the non-compact part as $Y^{\bar{a}\dot{a}\dot{b}\dot{c}} E_{\bar{a}\dot{a}\dot{b}\dot{c}}$. The coefficients satisfy symplectic reality conditions, and the coefficients for the commutators of two algebra elements are

$$
\begin{aligned}
[M, M']^{\dot{a}}{}_{\dot{b}} &= (M^{\dot{a}}{}_{\dot{c}} M'^{\dot{c}}{}_{\dot{b}} - M'^{\dot{a}}{}_{\dot{c}} M^{\dot{c}}{}_{\dot{b}}) \\
[M, Y]^{\bar{a}\dot{a}\dot{b}\dot{c}} &= 3 \cdot Y^{\bar{a}\dot{d}(\dot{a}\dot{b}} M^{\dot{c})}{}_{\dot{d}} \\
[Y, Y']^{\dot{a}}{}_{\dot{b}} &= (Y'^{\bar{a}\dot{a}\dot{c}\dot{d}} Y_{\bar{a}\dot{b}\dot{c}\dot{d}} - Y^{\bar{a}\dot{a}\dot{c}\dot{d}} Y'_{\bar{a}\dot{b}\dot{c}\dot{d}}) \\
[M, N] &= 0 \\
[N, N']^{\bar{a}}{}_{\bar{b}} &= (N^{\bar{a}}{}_{\bar{c}} N'^{\bar{c}}{}_{\bar{b}} - N'^{\bar{a}}{}_{\bar{c}} N^{\bar{c}}{}_{\bar{b}}) \\
[N, Y]^{\bar{a}\dot{a}\dot{b}\dot{c}} &= N^{\bar{a}}{}_{\bar{c}} Y^{\bar{c}\dot{a}\dot{b}\dot{c}} \\
[Y, Y']^{\bar{a}}{}_{\bar{b}} &= (Y'^{\bar{a}\dot{a}\dot{b}\dot{c}} Y_{\bar{b}\dot{a}\dot{b}\dot{c}} - Y^{\bar{a}\dot{a}\dot{b}\dot{c}} Y'_{\bar{b}\dot{a}\dot{b}\dot{c}}).
\end{aligned} \tag{2.19}
$$

This is worked out in some detail in appendix B. To ensure consistency, the fields

[6] For an introduction, see for example [136] or [61, Ch. 12]; a number of relevant formulae can be found in appendix A.2.

inherit the symplectic reality condition,

$$\begin{aligned}(\varphi^*)_{\bar a\,\dot a\dot b\dot c} &= -\varepsilon_{\bar a\bar b}\varepsilon_{\dot a\dot d}\varepsilon_{\dot b\dot e}\varepsilon_{\dot c\dot f}\varphi^{\bar b\,\dot d\dot e\dot f} =: -\varphi_{\bar a\,\dot a\dot b\dot c},\\ (\chi^*)_{i\,\dot a\dot b\dot c} &= -\varepsilon_{ij}\varepsilon_{\dot a\dot d}\varepsilon_{\dot b\dot e}\varepsilon_{\dot c\dot f}\chi^{j\,\dot d\dot e\dot f} =: -\chi_{i\,\dot a\dot b\dot c},\\ (\Psi^*)_{i\bar a} &= -\varepsilon_{ij}\varepsilon_{\bar a\bar b}\Psi^{j\bar b} \quad \text{(ditto for } \epsilon^{i\bar a}). \end{aligned} \tag{2.20}$$

With these algebra relations, one can construct the transformation behaviour of the fields under the local transformations of the $\mathfrak{so}(4) = \mathfrak{so}(3)_2 + \mathfrak{so}(3)_B$. Namely, let us demand that algebra-valued fields X and a transformation generated by the algebra element Y lead to a variation $\delta_Y X = [X, Y]$, while, more generally and true on the non-algebraic fields as well, demanding two $\mathfrak{so}(4)$-variations to satisfy $[\delta_Y, \delta_X] = \delta_{[Y,X]}$. The result is the transformation behaviour

$$\begin{aligned}(\delta\varphi)^{\bar a\,\dot a\dot b\dot c} &= -3\,\varphi^{\bar a\dot d(\dot a\dot b}M^{\dot c)}{}_{\dot d} - N^{\bar a}{}_{\bar b}\varphi^{\bar b\,\dot a\dot b\dot c}\\ (\delta\Psi)^{i\bar a} &= -N^{\bar a}{}_{\bar b}\Psi^{i\bar b}\\ (\delta\chi)^{i\,\dot a\dot b\dot c} &= -3\,\chi^{i\dot d(\dot a\dot b}M^{\dot c)}{}_{\dot d}. \end{aligned} \tag{2.21}$$

In (2.5), we have defined the composite objects P_μ and Q_μ. The field P_μ has the same index structure and transformation behaviour and obeys the same symplectic reality condition as $\varphi^{\bar a\dot a\dot b\dot c}$. Of the field Q_μ, there are two parts, corresponding to the $(\mathbf{1},\mathbf{3})$ and the $(\mathbf{3},\mathbf{1})$, respectively. In section 2.1, we have learned that Q_μ acts as the connection of the local H-symmetry. In the case we're dealing with here, there are two connections, one for $\mathfrak{so}(3)_B$, one for $\mathfrak{so}(3)_2$. With their help, let us define the action of the covariant derivative on the various fields as

$$\begin{aligned}(D_\mu(Q)P)^{\bar a\,\dot a\dot b\dot c} &= \partial_\mu P^{\bar a\dot a\dot b\dot c} + 3\,P^{\bar a\dot d(\dot a\dot b}Q_\mu{}^{\dot c)}{}_{\dot d} + Q_\mu{}^{\bar a}{}_{\bar b}P^{\bar b\dot a\dot b\dot c},\\ (D_\mu(Q)\Psi)^{i\bar a} &= \partial_\mu\Psi^{i\bar a} + Q_\mu{}^{\bar a}{}_{\bar b}\Psi^{i\bar b},\\ (D_\mu(Q)\chi)^{i\,\dot a\dot b\dot c} &= \partial_\mu\chi^{i\,\dot a\dot b\dot c} + 3\,\chi^{i\dot d(\dot a\dot b}Q_\mu{}^{\dot c)}{}_{\dot d} \end{aligned} \tag{2.22}$$

The $\mathfrak{so}(3)_B + \mathfrak{so}(3)_2$-covariant derivative obeys the usual Leibniz rule (making partial integration possible). It is readily modified to include a Lorentz-connection part acting on the spinors, $+1/4\cdot\gamma^{\alpha\beta}\omega_{\mu\alpha\beta}$ (with $\gamma^{\alpha\beta} = \gamma^{[\alpha}\gamma^{\beta]}$ and $\omega_{\mu\alpha\beta}$ the spin connection coefficients); the resulting derivative will be written as $D_\mu(\omega, Q)$.

Finally, we need to look at another batch of consequences of the symplectic reality condition (2.20), namely the symmetries and reality properties of spinor products. Fully contracted spinor products are symmetric and have Clifford conjugation as their adjoint, i.e.

$$(\bar\chi_{i\,\dot a\dot b\dot c}\gamma^{\mu_1\cdots\mu_m}\zeta^{i\,\dot a\dot b\dot c}) = (-)^{m(m+1)/2}(\bar\zeta_{i\,\dot a\dot b\dot c}\gamma^{\mu_1\cdots\mu_m}\chi^{i\,\dot a\dot b\dot c}) \tag{2.23}$$

and a corresponding relation for $(\overline{\Psi}_{i\bar{a}}\xi^{i\bar{a}})$. All such spinor products are purely imaginary. Using the symplectic reality condition, it is straightforward to show that with the contracted indices, the spinor products are indeed invariant under rigid symplectic transformations. With the isomorphism $\mathfrak{sp}(2) \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3)$, this corresponds to spinors ϕ^i transforming as spinors of a rigid $\mathfrak{so}(3)$.

For less-than-fully contracted products, there are slightly less simple symmetry relations which follow by resolving the product of two epsilon tensors into a sum of products of Kronecker deltas. Some such relations, which we shall need later on, are

$$(\overline{\Psi}_{j\bar{b}}\xi^{i\bar{a}}) = +(\overline{\xi}_{j\bar{b}}\Psi^{i\bar{a}}) + \delta^i_j\delta^{\bar{a}}_{\bar{b}}(\overline{\xi}_{k\bar{c}}\Psi^{k\bar{c}}) - \delta^{\bar{a}}_{\bar{b}}(\overline{\xi}_{j\bar{c}}\Psi^{i\bar{c}}) - \delta^i_j(\overline{\xi}_{k\bar{b}}\Psi^{k\bar{a}}), \tag{2.24}$$

$$(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^{\mu_1\cdots\mu_m}\zeta^{i\dot{a}\dot{b}\dot{d}}) = (-)^{m(m+1)/2}\left[\delta^{\dot{d}}_{\dot{c}}(\overline{\zeta}_{i\dot{a}\dot{b}\dot{e}}\gamma^{\mu_1\cdots\mu_m}\chi^{i\dot{a}\dot{b}\dot{e}}) - (\overline{\zeta}_{i\dot{a}\dot{b}\dot{c}}\gamma^{\mu_1\cdots\mu_m}\chi^{i\dot{a}\dot{b}\dot{d}})\right] \tag{2.25}$$

and

$$(\bar{\chi}_{i\,\dot{a}\dot{b}\dot{c}}\gamma^{\mu_1\cdots\mu_m}\zeta^{i\bar{b}}) = (-)^{m(m+1)/2}\varepsilon^{\bar{b}\bar{c}}\varepsilon_{\dot{a}\dot{d}}\varepsilon_{\dot{b}\dot{e}}\varepsilon_{\dot{c}\dot{f}}(\bar{\zeta}_{i\bar{c}}\gamma^{\mu_1\cdots\mu_m}\chi^{i\,\dot{d}\dot{e}\dot{f}}). \tag{2.26}$$

2.4 Lagrangian and supersymmetry variations: Construction

Next, let us formulate a core Lagrangian, consisting of kinetic terms only, and an ansatz for the supersymmetry variations. Augmenting the Lagrangian and the supersymmetry variations as necessary, we will then arrive at a modified Lagrangian that, up to fermionic terms cubic or higher, is supersymmetry invariant.

Let us start by defining the supersymmetry variations along the lines already laid down in (2.18). Keeping in mind that the scalar Lagrangian will involve the P_μ instead of the scalars φ themselves, one needs to amend the ansatz (2.18) accordingly, and, taking into consideration reality and proper dimensionality as well as introducing a real and dimensionless constant μ_1, arrives at

$$\begin{aligned}
\delta_S e_\mu{}^\alpha &= i\kappa^2(\overline{\epsilon}_{i\bar{a}}\gamma^\alpha\Psi_\mu^{i\bar{a}}) \\
\delta_S \Psi_\mu{}^{i\bar{a}} &= \epsilon_1(D_\mu(\omega, Q)\epsilon)^{i\bar{a}} \\
\delta_S \chi^{i\dot{a}\dot{b}\dot{c}} &= \mu_1 i\gamma^\mu(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}\varepsilon_{\bar{a}\bar{b}}\epsilon^{i\bar{b}},
\end{aligned} \tag{2.27}$$

with $\epsilon^{i\bar{a}}$ the (dimensionless) supersymmetry parameter, and with $\epsilon_1 = \pm 1$. Note that there are, so far, more free parameters than strictly necessary. By rescaling the gravitino and the fermion field χ, we could get rid of the sign ϵ_1 as well as of the parameter μ_1. My reason for keeping those two parameters is to facilitate the comparison of the this three-dimensional model with the dimensionally reduced five-dimensional supergravity. For the linearized supersymmetry variation of the scalar fields $\varphi^{\bar{a}\dot{a}\dot{b}\dot{c}}$, we have

$$\delta_{SL}\varphi^{\bar{a}\dot{a}\dot{b}\dot{c}} = \mu_2\kappa^2\varepsilon^{\bar{a}\bar{b}}(\overline{\epsilon}_{i\bar{b}}\chi^{i\dot{a}\dot{b}\dot{c}}), \tag{2.28}$$

with μ_2 yet another real, dimensionless constant.

The first ingredient of the Lagrangian is the pure supergravity part. Let us choose conventions with $c = \hbar = 1$, with [mass] as the basic dimension, with the vielbein dimensionless, and with Newton's constant κ^2 of the dimension of a length. With minimal modifications, namely adding the proper indices to the gravitino and making the derivative $SO(3)_B \times SO(3)_2$- as well as Lorentz covariant, the pure supergravity Lagrangian reads

$$-\frac{1}{4\kappa^2}\, e\mathcal{R} \;+\; \frac{i}{2}\epsilon_1 e(\overline{\Psi}_{\mu i\bar{a}}\gamma^{\mu\nu\rho}D_\nu(\omega, Q)\Psi_\rho^{i\bar{a}}), \tag{2.29}$$

where the relative coefficients have already been chosen with the cancellations of supersymmetry variations of pure supergravity in mind.

Next, for the scalars that form the coset model. The kinetic building block P_μ contains group elements and a derivative; keeping the group elements dimensionless, it has thus the dimension of an inverse length, $[P_\mu] = 1$. We have already seen that a "kinetic term" (in scare quotes, as it contains certain interactions, as well) can be formed using the Killing form κ, as $\kappa(P_\mu, P^\mu)$. Using that part of the (block-diagonal) Killing form acting on algebra elements in the $(\mathbf{4}, \mathbf{2})$, which is given as (B.9) in appendix B, adjusting for proper dimensionality and reality, and adding a real, dimensionless constant λ_1 to be fixed later, we obtain

$$\mathcal{L}_\varphi = \frac{\lambda_1}{\kappa^2} e g^{\mu\nu}(P_\mu)_{\bar{a}\dot{a}\dot{b}\dot{c}}(P_\nu)^{\bar{a}\dot{a}\dot{b}\dot{c}}. \tag{2.30}$$

Finally, we add a kinetic term for the matter fermions. Following Occam's dictum and avoiding use of the dimensionful constant κ^2, we let their dimension be $[\chi] = 1$; keeping the reality condition in mind and introducing one more real, dimensionless constant, λ_2, this term is

$$\mathcal{L}_\chi = i\lambda_2 e(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\mu D_\mu(\omega, Q)\chi^{i\dot{a}\dot{b}\dot{c}}). \tag{2.31}$$

These four terms constitute the manifestly $SO(3)_B \times SO(3)_2$-invariant core of our model's Lagrangian.

Next, let us proceed to check supersymmetry invariance in the 1.5 order formalism (i.e. using the equation of motion of the spin connection, but not of the other fields). First of all, note that the supersymmetry transformations of both P_μ and Q_μ, abstractly given as

$$\begin{aligned} \delta_S Q_\mu &= D_\mu(Q)\Sigma_S + [P_\mu, \delta_{SL}\varphi], \\ \delta_S P_\mu &= D_\mu(Q)\delta_{SL}\varphi + [P_\mu, \Sigma_S] \end{aligned}$$

in section 2.1, depend on the highly non-trivial entity Σ_S, an $\mathfrak{so}(4)$ transformation matrix defined by $\Sigma_S := -\tanh(1/2\ \mathrm{ad}_\varphi)\,\varphi$ and thus highly nonlinear in the scalar fields. A time-honoured and most economical way of dealing this is as follows. Note that what Σ_S contributes to both $\delta_S Q_\mu$ and $\delta_S P_\mu$ has the form of an infinitesimal local $SO(4)$ rotation. If we but add corresponding rotation terms to all the other supersymmetry variations, their total effect will be that of a local $SO(4)$ transformation on all fields – which will leave the Lagrangian invariant.

Thus, the ansatz for the supersymmetry variations is amended as follows:

$$\begin{aligned}
\delta_S \Psi_\mu{}^{i\bar{a}} &= \epsilon_1 (D_\mu(\omega, Q)\epsilon)^{i\bar{a}} - (\Sigma_S)^{\bar{a}}{}_{\bar{b}} \Psi^{i\bar{b}} \\
\delta_S \chi^{i\dot{a}\dot{b}\dot{c}} &= \mu_1 i\gamma^\mu (P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}} \varepsilon_{\bar{a}\bar{b}} \epsilon^{i\bar{b}} - 3\chi^{i\dot{d}(\dot{a}\dot{b}} (\Sigma_S)^{\dot{c})}{}_{\dot{d}}.
\end{aligned} \tag{2.32}$$

The variation of the $SO(4)$-singlet $e_\mu{}^\alpha$ is left unchanged. Henceforth, we need not bother about the contribution of Σ_S any more, as long as in amending the Lagrangian, we keep all additional terms $SO(4)$-invariant.

Next, for a look at the supergravity part (2.29). Inserting the variations (2.27), the usual cancellations for pure three-dimensional supergravity take place; schematically: in the Rarita-Schwinger term, variation of the gravitino introduces another covariant derivative, which can be partially integrated, leading to a commutator of two covariant derivatives $D(\omega)$ that can be rewritten in terms of the curvature to cancel the variation of the Einstein-Hilbert term. However, there is one difference: In the variation of the Rarita-Schwinger part, there is now a term containing $[D_\mu(\omega, Q), D_\nu(\omega, Q)]\Psi$ instead of simply $[D_\mu(\omega), D_\nu(\omega)]\Psi$ (as in the case of pure $\mathcal{N} = 1$ supergravity). However, from the definition of the covariant derivative $D(\omega, Q)$, one can derive the fact that $[D_\mu(\omega, Q), D_\nu(\omega, Q)]\Psi = [D_\mu(\omega), D_\nu(\omega)]\Psi + [D_\mu(Q), D_\nu(Q)]\Psi$. Thus, the cancellation of the Einstein-Hilbert variation goes through as before, but there is a remnant[7]

$$\delta\mathcal{L}_{sugra,rest} = \underbrace{\frac{i}{2}(e\epsilon_3 \varepsilon^{\mu\nu\rho})(\bar{\epsilon}_{i\bar{a}} [D_\mu(Q), D_\nu(Q)] \Psi_\rho^{i\bar{a}})}_{\boxed{\text{A}}}. \tag{2.33}$$

The next step is to vary $\mathcal{L}_\varphi$. Performing one partial integration, using

$$\delta_S e = -ie\kappa^2 (\overline{\Psi}_{\varrho i\bar{a}} \gamma^\varrho \epsilon^{i\bar{a}}) \quad \text{and} \quad \delta_S g^{\mu\nu} = 2i\kappa^2 (\overline{\Psi}^{(\mu}_{i\bar{a}} \gamma^{\nu)}\, \epsilon^{i\bar{a}})$$

[7] In the conventions used here, the curved-index epsilon symbol $\varepsilon^{\mu\nu\rho}$ is defined by applying vielbeins and metric coefficients to the all-lower flat-index epsilon symbol $\varepsilon_{\alpha\beta\delta}$ which, in its turn, is defined by $\varepsilon_{123} = +1$.

and discarding higher fermionic terms, the result is

$$\delta_S\mathcal{L}_\varphi = \underbrace{-i\lambda_1 e[(\overline{\Psi}_{\varrho i\bar{a}}\gamma^\varrho\epsilon^{i\bar{a}})g^{\mu\nu} - 2(\overline{\Psi}^{(\mu}_{i\bar{a}}\gamma^{\nu)}\epsilon^{i\bar{a}})](P_\mu)_{\bar{b}\dot{a}\dot{b}\dot{c}}(P_\nu)^{\bar{b}\dot{a}\dot{b}\dot{c}}}_{\boxed{\text{B}}}$$

$$\underbrace{-2\lambda_1\mu_2 e g^{\mu\nu}(D_\mu(Q)P_\nu)_{\bar{a}\dot{a}\dot{b}\dot{c}}\varepsilon^{\bar{a}\bar{b}}(\overline{\epsilon}_{i\bar{b}}\chi^{i\dot{a}\dot{b}\dot{c}})}_{\boxed{\text{C}}}$$

$$+\underbrace{2\lambda_1\mu_2\varepsilon^{\bar{a}\bar{b}}g^{\nu\rho}e_\gamma{}^\mu(D_{(\nu}(\omega)e_{\rho)}{}^\gamma)(P_\mu)_{\bar{a}\dot{a}\dot{b}\dot{c}}(\bar{\epsilon}_{i\bar{b}}\chi^{i\dot{a}\dot{b}\dot{c}})}_{\boxed{\text{D}}} \tag{2.34}$$

The variation of the matter fermions' kinetic term is

$$\delta_S\mathcal{L}_\chi = \underbrace{-2e\lambda_2\mu_1(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\varrho\gamma^\mu D_\varrho(\omega,Q)\epsilon^{i\bar{b}})(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}\varepsilon_{\bar{a}\bar{b}}}_{\boxed{\text{E}}}$$

$$\underbrace{-2e\lambda_2\mu_1(\overline{\epsilon}_{i\bar{b}}\chi^{i\dot{a}\dot{b}\dot{c}})\varepsilon^{\bar{a}\bar{b}}g^{\mu\nu}(D_\mu(Q)P_\nu)_{\bar{a}\dot{a}\dot{b}\dot{c}}}_{\boxed{\text{F}}},$$

$$\underbrace{+2\lambda_2\mu_1\varepsilon^{\bar{a}\bar{b}}g^{\nu\rho}e_\gamma{}^\mu(D_{(\nu}(\omega)e_{\rho)}{}^\gamma)(P_\mu)_{\bar{a}\dot{a}\dot{b}\dot{c}}(\bar{\epsilon}_{i\bar{b}}\chi^{i\dot{a}\dot{b}\dot{c}})}_{\boxed{\text{G}}} \tag{2.35}$$

where partial integration, the spinor identity (2.26), and the fact that, by (2.14) have been used, we have $D_\mu(Q)P_\nu = D_\nu(Q)P_\mu$. With these terms, a cancellation $\boxed{\text{C}} + \boxed{\text{F}} = 0$ is possible as long as we choose μ_2 so that

$$\lambda_2\mu_1 = -\lambda_1\mu_2.$$

Another consequence of this choice is the cancellation $\boxed{\text{D}} + \boxed{\text{G}} = 0$. Term $\boxed{\text{E}}$ is not so easily cancelled – none of the terms we have so far contains the proper combination of matter fermions and $D\epsilon$. To achieve cancellation, let us add a Noether term to the Lagrangian, namely

$$\mathcal{L}_N = 2e\lambda_2\mu_1\epsilon_1(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\varrho\gamma^\mu\Psi^{i\bar{b}}_\varrho)(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}\varepsilon_{\bar{a}\bar{b}}. \tag{2.36}$$

Varying $\Psi^{i\bar{b}}_\varrho$ in this term will exactly cancel $\boxed{\text{E}}$. However, supersymmetry variation of the matter fermion χ will give an additional contribution

$$\underbrace{2\lambda_2\mu_1^2\epsilon_1(e\epsilon_3\varepsilon^{\mu\nu\varrho})(\overline{\Psi}_{\varrho i\bar{a}}\epsilon^{i\bar{d}})(P_\nu)_{\bar{d}\dot{a}\dot{b}\dot{c}}(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}}_{\boxed{\text{H}}}$$

$$+\underbrace{2\lambda_2\mu_1^2\epsilon_1 e[2(\overline{\Psi}^{(\mu}_{i\bar{a}}\gamma^{\nu)}\epsilon^{i\bar{d}}) - (\overline{\Psi}_{\varrho i\bar{a}}\gamma^\rho\epsilon^{i\bar{d}})g^{\mu\nu}](P_\nu)_{\bar{d}\dot{a}\dot{b}\dot{c}}(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}}_{\boxed{\text{I}}}$$

plus higher fermionic terms. Noting that, for any tensor $T^{(\mu\nu)}$ symmetric in μ and ν, the relation

$$T^{(\mu\nu)}(P_\nu)_{\bar{d}\dot{a}\dot{b}\dot{c}}(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}} = +\frac{1}{2}T^{(\mu\nu)}\delta^{\bar{a}}_{\bar{b}}(P_\nu)_{\bar{c}\dot{a}\dot{b}\dot{c}}(P_\mu)^{\bar{c}\dot{a}\dot{b}\dot{c}}$$

holds, it can be seen that $\boxed{\text{I}} + \boxed{\text{B}} = 0$ if

$$\lambda_1 = -\lambda_2\mu_1^2\epsilon_1.$$

Presently, let us go back to $\boxed{\text{A}}$. By (2.13), there is a relation $[D_\mu(Q), D_\nu(Q)] = -[P_\mu, P_\nu]$. Using the algebra relations (2.19) and the transformation behaviour of $\Psi^{i\bar{a}}$ under an action with $[P_\mu, P_\nu]$ from the left (just as Q_μ, in the covariant derivative, acts from the left) it can be see that

$$([P_\mu, P_\nu]\Psi_\rho)^{i\bar{a}} = ((P_\mu)_{\bar{b}\dot{a}\dot{b}\dot{c}}(P_\nu)^{\bar{a}\dot{a}\dot{b}\dot{c}} - (P_\nu)_{\bar{b}\dot{a}\dot{b}\dot{c}}(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}})\Psi_\rho^{i\bar{b}}. \tag{2.37}$$

Using this and the spinor symmetry relation (2.24), it can be seen that $\boxed{\text{A}} + \boxed{\text{H}} = 0$ if

$$2\lambda_2\mu_1^2\epsilon_1 = 1.$$

With this, and up to higher fermionic terms, the Lagrangian is indeed supersymmetry invariant.

Taken together, this fixes all the new constants that had been introduced (except, of course, ϵ_1 and μ_1 which had deliberately been introduced as free parameters).

As a quick check, it is possible to calculate the easily obtainable parts of the supersymmetry algebra, namely the commutator of two supersymmetry variations on bosonic fields (on the fermionic side, being as we are in an on-shell supersymmetric model, we would have to contend with equations of motion, as well). The first is the commutator of supersymmetry transformations on the vielbein ${e_\mu}^\alpha$,

$$[\delta_{S_1}, \delta_{S_2}]{e_\mu}^\alpha = \left(\delta_{gct}(\xi^\nu) + \delta_L(\xi^\nu {\omega_\nu}^\alpha{}_\beta)\right){e_\mu}^\alpha,$$

where $\xi^\nu := i\kappa^2(\bar{\epsilon}_{2i\bar{a}}\gamma^\nu\epsilon_1^{i\bar{a}})$. Here, $\delta_{gct}(\xi^\nu)$ stands for an infinitesimal general coordinate transformation[8] with parameter ξ^ν, and $\delta_L(\xi^\nu{\omega_\nu}^\alpha)$ for an infinitesimal Lorentz transformation with parameter $\xi^\nu{\omega_\nu}^\alpha$ acting on the flat index α of the vielbein. This is the expected result.

The self-same commutator, applied to $P_\mu^{\bar{a}\dot{a}\dot{b}\dot{c}}$, gives

$$[\delta_{S_1}, \delta_{S_2}]P_\mu^{\bar{a}\dot{a}\dot{b}\dot{c}} = \epsilon_1\mu_1\mu_2\left(\delta_{gct}(\xi^\nu) - \delta_{so(3)_B}(\xi^\nu {Q_{\nu\dot{a}}}^{\dot{b}}) - \delta_{so(3)_2}(\xi^\nu {Q_\nu}^{\bar{a}}{}_{\bar{b}})\right)P_\mu^{\bar{a}\dot{a}\dot{b}\dot{c}},$$

a combination of general coordinate and two $\mathfrak{so}(3)$-transformations that again involve the parameter ξ^ν. As, by supersymmetry invariance, we already had found $\epsilon_1\mu_1\mu_2 = 1$, this is compatible with the commutator on the vielbein and concludes the quick check.

[8]Such transformations, the local generalizations of the translations $\xi^\mu\partial_\mu$ that are the standard supersymmetry-commutator in global supersymmetry, act on fields via the Lie derivative $\mathcal{L}_\xi$; on form fields like ${e_\mu}^\alpha$, it is $(\mathcal{L}_\xi e^\alpha)_\mu = \xi^\nu(\partial_\nu {e_\mu}^\alpha + {e_\nu}^\alpha(\partial_\mu\xi^\nu)$.

2.5 Lagrangian and supersymmetry variations: Results

Let us summarize the results of the previous section, eliminating those constants that have been fixed by the requirement of supersymmetry invariance. All in all, we have found the Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; e\Big\{ &-\frac{1}{4\kappa^2}\mathcal{R} + \frac{i}{2}\epsilon_1(\overline{\Psi}_{\mu i\bar{a}}\gamma^{\mu\nu\rho}D_\nu(\omega,Q)\Psi_\rho^{i\bar{a}}) - \frac{1}{2\kappa^2}g^{\mu\nu}(P_\mu)_{\bar{a}\dot{a}\dot{b}\dot{c}}(P_\nu)^{\bar{a}\dot{a}\dot{b}\dot{c}} \\ &+\frac{i}{2}\left(\frac{\epsilon_1}{\mu_1^2}\right)(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\mu D_\mu(\omega,Q)\chi^{i\dot{a}\dot{b}\dot{c}}) + \frac{1}{\mu_1}(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\rho\gamma^\mu\Psi_\rho^{i\bar{b}})(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}\varepsilon_{\bar{a}\bar{b}}\Big\} \quad (2.38)\end{aligned}$$

which, excluding quartic and higher terms, we have checked to be invariant under the supersymmetry variations

$$\begin{aligned}\delta_S e_\mu{}^\alpha &= i\kappa^2(\bar{\epsilon}_{i\bar{a}}\gamma^\alpha\Psi_\mu^{i\bar{a}}) \\ \delta_S\Psi_\mu{}^{i\bar{a}} &= \epsilon_1(D_\mu(\omega,Q)\epsilon)^{i\bar{a}} - (\Sigma_S)^{\bar{a}}{}_{\bar{b}}\Psi^{i\bar{b}} \\ \delta_S\chi^{i\dot{a}\dot{b}\dot{c}} &= \mu_1 i\gamma^\mu(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}\varepsilon_{\bar{a}\bar{b}}\epsilon^{i\bar{b}} - 3\,\chi^{i\dot{d}(\dot{a}\dot{b}}(\Sigma_S)^{\dot{c})}{}_{\dot{d}} \\ \delta_S(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}} &= \frac{\epsilon_1\kappa^2}{\mu_1}\varepsilon^{\bar{a}\bar{b}}D_\mu(Q)(\bar{\epsilon}_{i\bar{b}}\chi^{i\dot{a}\dot{b}\dot{c}}) - 3\,(P_\mu)^{\bar{a}\dot{d}(\dot{a}\dot{b}}(\Sigma_S)^{\dot{c})}{}_{\dot{d}} \\ &\qquad - (\Sigma_S)^{\bar{a}}{}_{\bar{b}}(P_\mu)^{\bar{b}\dot{a}\dot{b}\dot{c}} \\ \delta_S(Q_\mu)^{\dot{a}}{}_{\dot{b}} &= D_\mu(Q)\Sigma_S{}^{\dot{a}}{}_{\dot{b}} + \frac{\epsilon_1\kappa^2}{\mu_1}(P_\mu)_{\bar{a}\dot{f}\dot{c}\dot{d}}(2\delta^{\dot{a}}_{\dot{e}}\delta^{\dot{f}}_{\dot{b}} - \delta^{\dot{a}}_{\dot{b}}\delta^{\dot{f}}_{\dot{e}})\varepsilon^{\bar{a}\bar{b}}(\bar{\epsilon}_{i\bar{b}}\chi^{i\dot{e}\dot{c}\dot{d}}) \\ \delta_S(Q_\mu)^{\bar{a}}{}_{\bar{b}} &= D_\mu(Q)\Sigma_S{}^{\bar{a}}{}_{\bar{b}} + \frac{\epsilon_1\kappa^2}{\mu_1}(P_\mu)_{\bar{e}\dot{a}\dot{b}\dot{c}}(2\delta^{\bar{e}}_{\bar{b}}\varepsilon^{\bar{a}\bar{d}} - \delta^{\bar{a}}_{\bar{b}}\varepsilon^{\bar{e}\bar{d}})(\bar{\epsilon}_{i\bar{d}}\chi^{i\dot{a}\dot{b}\dot{c}}), \quad (2.39)\end{aligned}$$

with Σ_S the highly non-linear expression

$$\Sigma_S = -\tanh(1/2\cdot \mathrm{ad}_{\varphi})\,\varphi.$$

Chapter 3

Dimensional splits for supergravities

Dimensional reduction is defined as a compactification on a hypertorus, followed by a Fourier expansion of fields (this yields some massive Kaluza-Klein fields), and a shrinking of the compactification radii to zero – effectively decoupling the massive degrees of freedom, and leaving the dimensionally reduced Lagrangian. In practice, there is a simpler way to get at that Lagrangian: dropping the fields' dependence on the coordinates of the dimensions of the "internal space", plus some canonical rescaling of fields and coupling constants to adjust their dimensions leads to the same result. In the preparation of this kind of dimensional reduction, it is useful to split all objects that have d-dimensional vector- or form indices $M, N, \ldots$ (including derivatives) into those indices referring to lower-dimensional space-time indices, and those "internal" indices referring to the dimensions that are to compactified away. A similar split can be applied to spinorial indices. This is what, in the context of this chapter, will be called "dimensional splitting". Given a dimensional split, it is easy to obtain dimensionally reduced expressions by dropping the dependence on all internal coordinates, in effect setting all derivatives with respect to internal space coordinates to zero (plus some implicit rescaling to give the fields back their proper dimension).

Dimensional reduction is a very old technique [75, 82] and, in the context of higher-dimensional supergravities and string theories and together with more complex compactifications, has long become the material of review articles or even textbooks[1]. The specific redefinitions and reductions of the fermionic parts of the Lagrangian needed in the search for hidden symmetries, and especially the "dimensional splits" in which the dependency on the internal coordinates is, in contrast to reduction, still present, have, however, been performed more on a case-by-case basis, as in [31, 40, 104] (albeit with [48] containing somewhat more general expressions at least for the supersymmetry variations). In this chapter, dimensional splits and, eventually, dimensional reductions for supergravities will be performed. The exposition aims at a certain degree of generality – for part of the analysis, the dimensions $d = d_1 + d_2$ will be left completely unspecified; for another part, the calculations will be restricted to one of three possi-

[1] E.g. [29] or, in the more general context of string theory, chapter 8 in [114] or chapter 14 in [61].

ble cases for the split of gamma matrices; finally, there is the fact that the derivation proceeds via dimensional splits, and the dependence on the internal coordinates is not dropped right away. The generalizations are all reasonably straightforward, and it is not claimed that the results represent a substantial contribution to the literature; rather, in the context of this thesis, they are meant both to serve as an introduction to the subject of dimensional splits and to pave the way for the particular dimensional reduction undertaken in the next chapter, their generality facilitating the comparison with other cases such as the diverse reductions of eleven-dimensional supergravity; in a more general context, it is hoped that their combination of generality and explicitness might prove helpful to those who have it in mind to construct similar models. In particular, the generality serves to show which parallels between the eleven-dimensional E_8 case and the five-dimensional G_2 arise from the target dimension alone[2].

The chapter is organized as follows. Throughout, the starting point will be a supergravity in d dimensions, and the performance of a dimensional split $d = d_1 + d_2$, viewing the original d-dimensional space-time as a product of a d_1-dimensional space-time and a d_2-dimensional "internal space". This involves splitting all entities such as the vielbein, the metric etc. into components with different combinations of indices belonging to d_1 and indices belonging to d_2 and, in order to make sense of the split terms in the context of the d_1-dimensional theory, necessitates a number of redefinitions. For the vielbein, the anholonomy coefficients and the connection coefficients, the split will be introduced in section 3.1. Section 3.2 reviews the splitting of typical bosonic terms in the Lagrangian, namely Einstein-Hilbert, Maxwell and Chern-Simons terms for the special case of dimensional reduction (i.e. dropping the dependence of fields on the internal coordinates). Section 3.3 deals with the consequences of the split for gamma matrices and spinorial entities. Next, there are some consequences if the supersymmetry relations in the d_1-dimensional theory are to retain their usual form. Section 3.4 explores these as far as the supersymmetry relations of the erstwhile d-dimensional vielbein are concerned. Section 3.6 deals with the reduction of the gravitino kinetic term, and the remaining sections 3.7 and 3.8 with the supersymmetry variation of the new gravitino and spin-1/2 fields that result from the split of the original gravitino.

3.1 Splitting the vielbein

Here and in the following, Roman capitals denote $d = d_1 + d_2$-dimensional indices, flat ones from the beginning, curved ones from the middle of the alphabet. These indices are then split into d_1- and d_2-indices: The former are denoted by Greek letters, the latter (also called the "internal indices") by small Roman letters. The d_1-dimensional

[2] For some of the more voluminous calculation in this and the following chapters, I have used the low-level computer algebra program FORM, developed by Jos Vermaseren [135]. It allows the manipulation of simple (scalar, tensorial...) objects and thus came in useful for keeping track of a multitude of components arising from dimensional reduction or for checks of the algebraic results on the $G_{2(+2)}$, using explicit sets of generators.

space-time will often be referred to, in an abbreviated manner, as “the space-time”, while the d_2-dimensional space will be called the “internal space”.

The first object to split is the d-dimensional vielbein $E_M{}^A$. The procedure to do this has been worked out long ago in the context of dimensional reduction of pure gravity [21, 22, 31, 30, 124]. A priori, there are four possible index combinations: An object with all space-time indices, $E_\mu{}^\alpha$, two different kinds of objects with mixed indices, $E_\mu{}^a$ and $E_m{}^\alpha$, and an object with two internal indices, $E_m{}^a$. It is, however, possible to simplify matters by choosing a special Lorentz group gauge for the d–dimensional vielbein, making $E_m{}^\alpha = 0$. The resulting object can be schematically written as

$$E_M{}^A = \begin{pmatrix} E_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}. \tag{3.1}$$

The principal off-diagonal object B_μ^n is called the Kaluza-Klein-vector; in the original Kaluza-Klein reduction $d_1 = 4$, $d_2 = 1$, aimed at unifying electromagnetism and gravity, it was supposed to play the role of the electromagnetic potential.

The goal is to make things look as natural as possible in d_1–dimensional space-time. In particular, there should be d_1–dimensional gravity, i.e. a term in the Lagrangian that looks like $\sqrt{|g^{(d_1)}|}\mathcal{R}^{(d_1)}$, with $g^{(d_1)}$ the determinant of the d_1–dimensional metric and $\mathcal{R}^{(d_1)}$ the d_1–dimensional curvature scalar. However, a simple block split does not achieve this. Suppose that, in (3.1) above, we had kept $E_\mu{}^\alpha$ as our d_1-dimensional vielbein, the d_1+d_2–dimensional Einstein-Hilbert part of the action would have given rise to one term that contained the proper curvature scalar, but this term would still have had the whole $d_1 + d_2$–dimensional determinant in front of it, and would look like $\sqrt{|g^{(d_1)}|}\sqrt{|g^{int}|}\mathcal{R}^{(d_1)}$, i.e. not Einstein gravity, but some kind of dilatonic gravity instead. With the exception of $d_1 = 2$, an especially problematic case that shall be excluded henceforth by a restriction to $d_1 > 2$, there is a remedy: In order to transform this into the proper form, one can make a Weyl rescaling of the d_1–dimensional part of the metric, namely of $E_\mu{}^\alpha$. A rescaling $E_\mu{}^\alpha \mapsto \Omega e'_\mu{}^\alpha$ corresponds to a rescaling of the metric $g_{\mu\nu} \mapsto \Omega^2 g'_{\mu\nu}$. This changes the term containing the d_1–dimensional curvature scalar: From the determinant, there is an extra factor Ω^{d_1}, from the curvature scalar, a factor Ω^{-2}. This cancels the tiresome powers of $\sqrt{|g^{int}|} = \det e_m{}^a =: \Delta$ in front of the Einstein-Hilbert term if the choice is $\Omega = \Delta^{-1/(d_1-2)}$, corresponding, in the above vielbein, to a substitution $E_\mu{}^\alpha = \Delta^s e'_\mu{}^\alpha$, with $s = -1/(d_1 - 2)$. With this substitution, the determinant E of the d-dimensional vielbein is seen to be $E = \Delta^{sd_1}\ e' \det e_m{}^a = \Delta^{sd_1+1}\ e' = \Delta^{2s}\ e'$, with e' the determinant of the rescaled d_1-dimensional vielbein. The inverse vielbein corresponding to the rescaled (3.1) is

$$E_A{}^M = \begin{pmatrix} \Delta^{-s}\ e'_\alpha{}^\mu & -\Delta^{-s} e'_\alpha{}^\mu B_\mu{}^m \\ 0 & e_a{}^m \end{pmatrix}, \tag{3.2}$$

with the new objects $e'_\alpha{}^\mu$ and $e_a{}^m$ defined in the usual way via $e'_\alpha{}^\mu\, e'_\mu{}^\beta = \delta_\alpha^\beta$ and $e_a{}^m\, e_m{}^b = \delta_a^b$.

In the dimensional split, there is a possibility of signature change – say, while the original space-time had signature $(+,-,\cdots,-)$, the d_1-dimensional space-time has signature $(-,+,\cdots,+)$, or the internal space-time signature $(+,\cdots,+)$. In consequence, the split of η^{AB} into flat partial metrics might involve a sign change either in the new space-time flat metric or the internal metric. Define the flat metrics for the split spaces as

$$\eta'^{\alpha\beta} = \zeta_1\, \eta^{\alpha\beta} \quad \text{and} \quad \bar{\eta}^{ab} = \zeta_2\, \eta^{ab}, \tag{3.3}$$

where $\zeta_1 = \pm 1$, $\zeta_2 = \pm 1$, with a minus sign indicating a change of signature. In addition, define the curved metrics for the split spaces as

$$\begin{aligned} g'^{\mu\nu} &= \eta'^{\alpha\beta} e'_\alpha{}^\mu e'_\beta{}^\nu = \zeta_1\, \eta^{\alpha\beta} e'_\alpha{}^\mu e'_\beta{}^\nu \\ \bar{g}^{mn} &= \bar{\eta}^{ab} e_a{}^m e_b{}^n = \zeta_2\, \eta^{ab} e_a{}^m e_b{}^n. \end{aligned} \tag{3.4}$$

Last, not least, one can use the vielbein and its inverse to reconstruct the metric and inverse metric in this special gauge. The results are

$$g^{MN} = \Delta^{-2s} \begin{pmatrix} \zeta_1\, g'^{\mu\nu} & -\zeta_1\, g'^{\mu\varrho} B_\varrho{}^n \\ -\zeta_1\, g'^{\nu\varrho} B_\varrho{}^m & \zeta_1\, g'^{\varrho\sigma} B_\varrho{}^m B_\sigma{}^n + \Delta^{2s}\zeta_2\, \bar{g}^{mn} \end{pmatrix} \tag{3.5}$$

and

$$g_{MN} = \begin{pmatrix} \Delta^{2s}\zeta_1\, g'_{\mu\nu} + \zeta_2\, \bar{g}_{qr} B_\mu{}^q B_\nu{}^r & \zeta_2\, B_\mu{}^r \bar{g}_{rn} \\ \zeta_2\, \bar{g}_{mr} B_\nu{}^r & \zeta_2\, \bar{g}_{mn} \end{pmatrix}. \tag{3.6}$$

In consequence of this split, one needs to exercise care in going from flat indices to curved indices (e.g. from a to m, or from α to μ), as well as when raising indices like m or μ – even an operation as seemingly innocuous as the simultaneous raising and lowering of indices (say, from a Northeast-to-Southwest to a Northwest-to-Southeast contraction) becomes non-trivial. For future reference, let us write out the transition rules. For a vector $v = v^M \partial_M = v^A E_A$, split as $v_M = (v_\mu, v_m)$ and $v_A = (v_\alpha, v_a)$, flat and curved components are related as

$$\begin{aligned} v^a &= (v^m + v^\mu B_\mu{}^m) e_m{}^a \\ v^\alpha &= \Delta^s v^\mu\, e'_\mu{}^\alpha \\ v^m &= v^a e_a{}^m - \Delta^{-s} v^\alpha e'_\alpha{}^\mu B_\mu{}^m = v^a e_a{}^m - v^\mu B_\mu{}^m \\ v^\mu &= \Delta^{-s}\, v^\alpha e'_\alpha{}^\mu, \end{aligned} \tag{3.7}$$

while for a one–form $\omega = \omega_M \mathrm{d}x^M = \omega_A E^A$, the corresponding relationships are

$$\begin{aligned} \omega_a &= e_a{}^m \omega_m \\ \omega_\alpha &= \Delta^{-s}\, e'_\alpha{}^\mu (\omega_\mu - B_\mu{}^m \omega_m) \\ \omega_m &= e_m{}^a \omega_a \\ \omega_\mu &= \Delta^s\, e'_\mu{}^\alpha \omega_\alpha + B_\mu{}^m e_m{}^a \omega_a = \Delta^s\, e'_\mu{}^\alpha \omega_\alpha + B_\mu{}^m \omega_m. \end{aligned} \tag{3.8}$$

Similarly, one must take care while raising or lowering indices of the same type, either with the flat or with the curved metric. For the flat metrics, one can pick up additional sign factors $\zeta_{1,2}$. For the curved metrics, using the expressions (3.6) and (3.5) for the metric and its inverse, the result is

$$\begin{aligned} v_m &= \zeta_2\, \bar{g}_{mn}(v^n + v^\nu B_\nu{}^n) \\ v^m &= \zeta_2\, \bar{g}^{mn} v_n - v^\nu B_\nu{}^m \\ v_\mu &= \zeta_1 \Delta^{2s} g'_{\mu\nu} v^\nu + B_\mu{}^m v_m \\ v^\mu &= \zeta_1 \Delta^{-2s} g'^{\mu\nu}(v_\nu - B_\nu{}^n v_n). \end{aligned} \tag{3.9}$$

As has been mentioned already, for curved indices, this has the consequence that Northeast-to-Southwest contraction is not the same as Northwest-to-Southeast.

Probably the simplest way to circumvent all these complications is the following: For each field to be split, choose one type of index – say, perform the split of a vector field v by splitting its lowered flat components, $v_A = (v_\alpha, v_a)$. After the split, choose the convention that any occurrence of v with a different index means a transformation using the vielbeine $e'_\mu{}^\alpha$, $e_m{}^a$ or their inverse, and the metrics $\eta'^{\alpha\beta}$, $\bar{\eta}^{ab}$, $g'^{\mu\nu}$ and $\bar{g}^{mn}$.

From the vielbein, on to related entities. The first are the anholonomy coefficients, defined by means of the Lie bracket for vector fields via $[E_A, E_B]^M = \Omega_{AB}{}^C\, E_C{}^M$ or, explicitly, via the vielbein as

$$\Omega_{AB}{}^C = -2\; E_{[A}{}^N\, E_{B]}{}^S(\partial_N\, E_S{}^C); \tag{3.10}$$

a measure of the vielbein's deviation from a coordinate ("holonomic") basis, whose basis vectors – partial derivatives – all commute. For convenience, define the derivative $\mathcal{D}_\mu := \Delta^s e'_\mu{}^\alpha E_\alpha{}^M \partial_M = \partial_\mu - B_\mu{}^m \partial_m$. Then the different anholonomy coefficients are

$$\begin{aligned} \Omega_{\alpha\beta}{}^\gamma &= -2\Delta^{-2s} e'_{[\alpha}{}^\mu e'_{\beta]}{}^\nu\, \mathcal{D}_\mu(\Delta^s\, e'_\nu{}^\gamma). \\ &= -2\Delta^{-s}\left[e'_{[\alpha}{}^\mu e'_{\beta]}{}^\nu (\mathcal{D}_\mu e'_\nu{}^\gamma) + s\, e'_{[\alpha}{}^\mu \delta^\gamma_{\beta]}(\mathcal{D}_\mu \ln\Delta)\right] \\ \Omega_{\alpha\beta}{}^c &= -2\Delta^{-2s} e'_{[\alpha}{}^\mu e'_{\beta]}{}^\nu\, \mathcal{D}_\mu(B_\nu{}^n) e_n{}^c \\ \Omega_{a\beta}{}^\gamma &= -\Delta^{-s} e_a{}^m e'_\beta{}^\nu \partial_m(\Delta^s\, e'_\nu{}^\gamma) \\ &= -e_a{}^m\left[e'_\beta{}^\nu(\partial_m e'_\nu{}^\gamma) + s(\partial_m \ln\Delta)\delta^\gamma_\beta\right] \\ &= -\Omega_{\beta a}{}^\gamma \\ \Omega_{ab}{}^\gamma &= 0 \end{aligned} \tag{3.11}$$

$$\begin{aligned} \Omega_{a\beta}{}^c &= \Delta^{-s}\, e'_\beta{}^\nu e_a{}^m \left[\mathcal{D}_\nu(e_m{}^c) - (\partial_m B_\nu{}^n) e_n{}^c\right] \\ &= -\Omega_{\beta a}{}^c \\ \Omega_{ab}{}^c &= -2 e_{[a}{}^m e_{b]}{}^n \partial_m(e_n{}^c). \end{aligned} \tag{3.12}$$

In the following, it will prove useful to divide each of those coefficients up into a part independent of Δ and a part that contains Δ and its derivatives. In order to do that, define Δ-independent expressions $\Omega'_{AB}{}^{C}$ by

$$\begin{aligned}
\Omega'_{\alpha\beta}{}^{\gamma} &:= -2e'_{[\alpha}{}^{\mu}e'_{\beta]}{}^{\nu}\,\mathcal{D}_{\mu}(e'_{\nu}{}^{\gamma}) \\
\Omega'_{\alpha\beta}{}^{c} &:= -2e'_{[\alpha}{}^{\mu}e'_{\beta]}{}^{\nu}\,\mathcal{D}_{\mu}(B_{\nu}{}^{n})e_{n}{}^{c} \\
\Omega'_{a\beta}{}^{\gamma} &:= -e_{a}{}^{m}e'_{\beta}{}^{\nu}\,\partial_{m}(e'_{\nu}{}^{\gamma}) \\
\Omega'_{a\beta}{}^{c} &:= e'_{\beta}{}^{\nu}e_{a}{}^{m}\left[(\mathcal{D}_{\nu}e_{m}{}^{c}) - (\partial_{m}B_{\nu}{}^{n})e_{n}{}^{c}\right] \\
\Omega'_{ab}{}^{c} &:= -2e_{[a}{}^{m}e_{b]}{}^{n}(\partial_{m}e_{n}{}^{c})
\end{aligned} \tag{3.13}$$

and by antisymmetry of their first two indices. With their help, it is possible to rewrite the anholonomy coefficients as

$$\begin{aligned}
\Omega_{\alpha\beta}{}^{\gamma} &= \Delta^{-s}\left[\Omega'_{\alpha\beta}{}^{\gamma} - 2se'_{[\alpha}{}^{\mu}\delta^{\gamma}_{\beta]}\,(\mathcal{D}_{\mu}\ln\Delta)\right] \\
\Omega_{\alpha\beta}{}^{c} &= \Delta^{-2s}\,\Omega'_{\alpha\beta}{}^{c} \\
\Omega_{a\beta}{}^{\gamma} &= \Omega'_{a\beta}{}^{\gamma} - s\,e_{a}{}^{m}(\partial_{m}\ln\Delta)\delta^{\gamma}_{\beta} = -\Omega_{\beta a}{}^{\gamma} \\
\Omega_{ab}{}^{\gamma} &= 0 \\
\Omega_{a\beta}{}^{c} &= \Delta^{-s}\,\Omega'_{a\beta}{}^{c} = -\Omega_{\beta a}{}^{c} \\
\Omega_{ab}{}^{c} &= \Omega'_{ab}{}^{c}.
\end{aligned} \tag{3.14}$$

Finally, to the spin connection which is needed to define Lorentz-covariant derivatives. By the usual vielbein postulate (demanding that the covariant derivative defined by the spin connection is equal to the generally covariant metric derivative, lifted into the flat tangent space by the vielbein), the relation

$$\omega_{MAB} = \frac{1}{2}E_{M}{}^{C}\left[\Omega_{ABC} - \Omega_{BCA} - \Omega_{CAB}\right] \;+\; K_{AMB} \tag{3.15}$$

holds, with K_{AMB} the components of the contorsion[3]. One can write down a split version of that part of the spin connection which does not depend on the contorsion – let us call it $\omega(E)$ – with the help of the anholonomy coefficients, defined above. The

[3] For the conventions used, see appendix A.2.

result is

$$\begin{aligned}
\omega_{m\alpha\beta}(E) &= \frac{1}{2}e_m{}^c(\zeta_2\Delta^{-2s}\Omega'_{\alpha\beta c} - 2\zeta_1\Omega'_{c[\alpha\beta]}) \\
\omega_{ma\beta}(E) &= -\zeta_2\Delta^{-s}\, e_m{}^c\Omega'_{\beta(ac)} \\
&= \frac{\zeta_2}{2}\Delta^{-s}e'_\beta{}^\nu e_a{}^n[(\mathcal{D}_\nu\bar{g}_{mn}) - 2\bar{g}_{p(m}(\partial_{n)}B_\nu{}^p)] \\
&= -\omega_{m\beta a}(E) \\
\omega_{mab}(E) &= \frac{1}{2}\zeta_2 e_m{}^c(\Omega'_{abc} - \Omega'_{bca} - \Omega'_{cab}) \\
\omega_{\mu\alpha\beta}(E) &= \zeta_1\omega'_{\mu\alpha\beta} + 2s\zeta_1 e'_{\mu[\alpha}e'_{\beta]}{}^\nu\,(\mathcal{D}_\nu \ln\Delta) + B_\mu{}^m\omega_{m\alpha\beta} \\
\omega_{\mu a\beta}(E) &= \frac{1}{2}\,e'_\mu{}^\gamma(2\Delta^s\zeta_1\Omega'_{a(\beta\gamma)} - \zeta_2\Delta^{-s}\Omega'_{\beta\gamma a}) \\
&\quad -\zeta_1 s\,\Delta^s e'_{\mu\beta}e_a{}^m(\partial_m\ln\Delta) + B_\mu{}^m\omega_{ma\beta} \\
&= -\omega_{\mu\beta a}(E) \\
\omega_{\mu ab}(E) &= B_\mu{}^m\omega_{mab} - \zeta_2 e'_\mu{}^\gamma\Omega'_{\gamma[ab]},
\end{aligned} \tag{3.16}$$

where $\omega'_{\mu\alpha\beta}$ is the "spin connection" obtained by using the primed omega coefficients (not quite, but nearly the anholonomy coefficients of the primed vielbein) in eq. (3.15) and lowering the third index of the $\Omega_{\alpha\beta}{}^\gamma$ with $\eta'_{\alpha\beta}$.

3.2 Dimensional reduction of bosonic terms: Einstein-Hilbert and Maxwell

The vielbein split accomplished, it is as good a time as any to recall some standard results about the dimensional reduction of two kinds of bosonic terms that we are bound to encounter in a supergravity Lagrangian – the ubiquitous Einstein-Hilbert term, and a Maxwell kinetic term [29]. Unlike in the fermionic sector, there is no satisfying way to rewrite the dimensionally-split bosonic terms, still depending on their internal coordinates, in a compact way. Thus, let us restrict ourselves right away to the case of dimensional reduction, where we will need the reduced terms as a point of comparison for the scalar Lagrangian derived from the coset model.

Reducing Einstein-Hilbert

For calculating the reduced Einstein-Hilbert action, it is convenient to postpone the Weyl rescaling to the very end of the calculation. Hence, for the moment, let us keep the $E_\mu{}^\alpha$ in the split vielbein, and calculate derived entities such as the spin coefficients in terms of these objects. Define the field strength for the Kaluza-Klein vector $B_\mu{}^n$ as

$$G^n_{\mu\nu} = \partial_\mu B_\nu{}^n - \partial_\nu B_\mu{}^n. \tag{3.17}$$

If, in line with the dimensional reduction program, the dependence on the internal coordinates is dropped (setting to zero all terms involving derivatives ∂_m), the non-zero coefficients of anholonomy are

$$\begin{aligned}
\Omega_{\alpha\beta}{}^{\gamma} &= -2E_{[\alpha}{}^{\mu}E_{\beta]}{}^{\nu}(\partial_\mu E_\nu{}^{\gamma}) \\
\Omega_{\alpha\beta}{}^{c} &= -E_\alpha{}^{\mu}E_\beta{}^{\nu}G_{\mu\nu}{}^{n}e_n^c \\
\Omega_{a\beta}{}^{c} &= E_\beta{}^{\mu}e_a{}^{n}(\partial_\mu e_n{}^{c}) = -\Omega_{\beta a}{}^{c}.
\end{aligned}$$

while $\Omega_{ab}{}^{\gamma} = \Omega_{a\beta}{}^{\gamma} = \Omega_{ab}{}^{c} = 0$. The most economic way of calculating the dimensional reduction for the Einstein-Hilbert term makes use of the fact that it can be expressed in terms of the anholonomy coefficients as[4]

$$E\mathcal{R} = \frac{1}{4}E\left[-\Omega_{ABC}\Omega^{ABC} + 2\,\Omega_{ABC}\Omega^{CAB} + 4\,\Omega_{AB}{}^{B}\Omega^{A}{}_{C}{}^{C}\right],$$

which holds up to a boundary term. Inserting the dimensionally reduced anholonomy coefficients derived above, the result is

$$\begin{aligned}
E\mathcal{R} = \Delta e^{(3)}\Big[&\zeta_1\,\mathcal{R}^{(3)} - \zeta_2\,\frac{1}{4}\,g'^{\mu\rho}g'^{\nu\sigma}\overline{g}_{mn}G^m_{\mu\nu}G^n_{\rho\sigma} \\
&+ \frac{1}{4}\zeta_1\,g'^{\mu\nu}(\overline{g}^{mk}\overline{g}^{nl} - \overline{g}^{ml}\overline{g}^{nk})(\partial_\mu\overline{g}_{mk})(\partial_\nu\overline{g}_{nl})\Big],
\end{aligned}$$

where $e^{(3)}$ and $\mathcal{R}^{(3)}$ are three-dimensional metric determinant and curvature scalar associated with the vielbein $E_\mu{}^{\alpha}$. The final term, the kinetic term of the internal metric, can be rewritten a bit to show its structure more clearly. Using the $\Delta = \det(e_m{}^{a})$, direct calculation shows that

$$\frac{1}{4}g'^{\mu\nu}(\overline{g}^{mk}\overline{g}^{nl} - \overline{g}^{ml}\overline{g}^{nk})(\partial_\mu\overline{g}_{mk}) = g'^{\mu\nu}(\partial_\mu\ln\Delta)(\partial_\nu\ln\Delta) + \frac{1}{4}g'^{\mu\nu}(\partial_\mu\overline{g}_{mn})(\partial_\nu\overline{g}^{mn}).$$

Then, it is time to reincorporate the Weyl rescaling. Substituting $E_\mu{}^{\alpha} = \Delta^s e'_\mu{}^{\alpha}$, we obtain the final result

$$\begin{aligned}
E\mathcal{R} = e'\Big[&\zeta_1\,\mathcal{R}' - \frac{\zeta_1}{(d_1-2)}\;g'^{\mu\nu}(\partial_\mu\ln\Delta)(\partial_\nu\ln\Delta) \\
&+ \frac{1}{4}\,\zeta_1\,g'^{\mu\nu}(\partial_\mu\overline{g}_{mn})(\partial_\nu\overline{g}^{mn}) - \frac{1}{4}\zeta_2\,\Delta^{-2s}\;g'^{\mu\rho}g'^{\nu\sigma}\,\overline{g}_{mn}G^m_{\mu\nu}G^n_{\rho\sigma}\Big]. \qquad (3.18)
\end{aligned}$$

The first of the terms vindicates the Weyl rescaling – it is the three-dimensional Einstein-Hilbert term involving the metric determinant e' and the curvature scalar $\mathcal{R}'$ associated with the vielbein $e'_\mu{}^{\alpha}$. The second and third term are kinetic terms for the $d_2(d_2-1)/2$ scalars, while the final term is a Maxwell kinetic term for the d_2 vector fields $B_\mu{}^{n}$, modified by the inclusion of the factor Δ^{-2s}.

[4]Cf. appendix A.2.

Splitting Maxwell

Next, for the split of the covariantized Maxwell Lagrangian $EF_{MN}g^{MR}g^{NS}F_{RS}$, with the usual definition $F_{MN} = \partial_M A_N - \partial_N A_M$ for the field strength. At the same time, the splitting procedure can serve as a model for the splitting of the Lagrangians of higher p-form fields.

Again, the split will depend on which incarnation of the A field – A_A or A_M – is considered to be fundamental. Let us follow the general recipe of splitting the flat components, $A_A = E_A{}^M A_M$, which leads to components

$$A_\alpha = E_\alpha{}^\mu (A_\mu - B_\mu{}^m A_m)$$
$$A_a = e_a{}^m A_m. \qquad (3.19)$$

The combination of A_μ and A_m that is part of these components suggests the definition of a field $A'_\mu := A_\mu - B_\mu{}^m A_m$ (which has the added advantage of being invariant under the Abelian symmetry associated with the d_2 Kaluza-Klein vectors). In the same manner, one can split the flat components F_{AB} of the field strength, noting that

$$F_{AB} = E_A{}^M(\partial_M A_B) - E_B{}^M(\partial_M A_A) - \Omega_{AB}{}^C A_C. \qquad (3.20)$$

The result is

$$\begin{aligned} F_{\alpha\beta} &= E_\alpha{}^\mu E_\beta{}^\nu (\partial_\mu A'_\nu - \partial_\nu A'_\mu + G^n_{\mu\nu} A_n) \\ F_{\alpha a} &= E_\alpha{}^\mu e_a{}^n (\partial_\mu A_n) \\ F_{ab} &= 0. \end{aligned} \qquad (3.21)$$

With these preparations, now for the splitting of the kinetic term $EF_{AB}F^{AB}$. Rescaling, as before, $E_\alpha{}^\mu = \Delta^{-s} e'_\alpha{}^\mu$ (which, for this kinetic term, makes a difference in E only) and defining $F'_{\mu\nu} = e'_\mu{}^\alpha e'_\nu{}^\beta F_{\alpha\beta}$, the result is

$$EF_{AB}F^{AB} = e'\Delta^{2s}F'_{\mu\nu}F'_{\mu\nu} + 2\,\zeta_1\zeta_2\, e'\, \overline{g}^{mn} g'^{\mu\nu}(\partial_\mu A_m)(\partial_\nu A_n),$$

a simple kinetic term for the scalars that are the internal vector components, a modified Maxwell term for the vector fields, reminiscent of the modified Maxwell term we had found for the Kaluza-Klein vectors. However, it must be noted that the newly defined field strength $F'_{\mu\nu}$ lacks one property that one is wont to associate with the field strengths of vector fields: Unlike its precursor F_{MN}, it does not satisfy a Bianchi identity of the form $\partial_{[\mu}F'_{\nu\rho]}$! In particular this means that whenever, for some reason or other, one needs to dualize such a vector field, it is necessary to introduce an auxiliary field $\tilde{F}_{\mu\nu} := \Delta^{2s}F'_{\mu\nu} - G^m_{\mu\nu}A_m = 2\partial_{[\mu}A'_{\nu]}$ which does satisfy a Bianchi identity.

3.3 Splitting Spinors and Gamma matrices

For vectors and tensors, the dimensional split is simply a matter of dividing up components. For spinors, matters are a bit more involved. Basically, there are three different cases, two of which are readily resolved. Let us start with the Clifford algebra of the d-dimensional theory – the algebra of the matrices Γ^A obeying the Clifford relation $\Gamma^A\Gamma^B + \Gamma^B\Gamma^A = 2\eta^{AB}$. There is a general result that has wide application in the classification and construction of Clifford algebras (e.g. [6, 112], and cf. chapter 5), namely that there is a simple general recipe to build a larger Clifford algebra as the tensor product of two smaller ones – as long as one of the smaller algebras has an even number of generators. This is readily seen: Denote the n_1 generators of one of the algebras by e_i, and the n_2 generators of the other (with n_2 even) by $\bar{e}_i$. Furthermore, define the usual "Clifford volume element" for the second algebra by $\bar{e}^v = \bar{e}_1\bar{e}_2\cdots\bar{e}_{n_2}$. As n_2 is even, the Clifford relation entails that $\bar{e}^v$ anticommutes with each of the generators $\bar{e}_i$. Hence, one can define $n_1 + n_2$ new generators $e_i \otimes \bar{e}^v$ and $\mathbb{1} \otimes \bar{e}_i$ which satisfy a Clifford algebra relation – different generators anticommute, and every generator squares to either plus or minus one (with the signature dependent on those of the two factor algebras).

This recipe is readily applied to the dimensional splitting of spinors. There is one case in which the simple tensor factoring fails, and which we shall leave aside, namely that of d even, and both d_1 and d_2 odd. It involves an extra tensor factor leading to an additional index [140].

Denote the space-time matrices by γ^α (with $\alpha = 1, \ldots, d_1$) and the matrices belonging to the internal space by $\hat{\Gamma}^a$ (with $a = 1, \ldots, d_2$). The first possibility is

$$\Gamma^A = \begin{cases} \gamma^\alpha \otimes \hat{\Gamma}^v \\ \mathbb{1} \otimes \hat{\Gamma}^a \end{cases} \tag{3.22}$$

with $\hat{\Gamma}$ appropriate gamma matrices for the "internal" Clifford algebra, $\hat{\Gamma}^v$ their volume element, and γ^α the gamma matrices for the "space-time" Clifford algebra. For this to work, d_2 has to be even, in order that $\hat{\Gamma}^v$ anticommute with all generators $\hat{\Gamma}^a$. In the following, I will call this "case A". For this case, define ϵ_v to be the sign of the square of $\hat{\Gamma}^v$, $(\hat{\Gamma}^v)^2 = \epsilon_v \mathbb{1}$. The split of Clifford matrices dictates the split of the flat metric η^{AB}, namely, with the notation (3.3), case A corresponds to $\zeta_2 = 1$ and $\zeta_1 = \epsilon_v$. For multi-index gamma matrices (defined as the antisymmetrized products of single-index gamma matrices, e.g. $\Gamma^{AB} := \Gamma^{[A}\Gamma^{B]}$), this means

$$\Gamma^{\alpha_1\cdots\alpha_r a_1\cdots a_s} = \gamma^{\alpha_1\cdots\alpha_r} \otimes (\hat{\Gamma}^v)^r\, \hat{\Gamma}^{a_1\cdots a_s}. \tag{3.23}$$

The other possibility is case B,

$$\Gamma^A = \begin{cases} \gamma^\alpha \otimes \ \mathbb{1} \\ \gamma^v \otimes \ \hat{\Gamma}^a \end{cases}, \tag{3.24}$$

which only works for d_1 even, in order for γ^v to anticommute with all generators γ^a. Here, define ϵ_v by $(\gamma^v)^2 = \epsilon_v \mathbb{1}$. A consequence of the split is $\zeta_2 = \epsilon_v$, while $\zeta_1 = 1$. For multi-index gamma matrices,

$$\Gamma^{\alpha_1\cdots\alpha_r a_1\cdots a_s} = \gamma^{\alpha_1\cdots\alpha_r} (\hat{\gamma}^v)^s \otimes \hat{\Gamma}^{a_1\cdots a_s}. \tag{3.25}$$

The tensor factoring of the gamma matrices implies the split for the spinors (or, more generally, pinors) they act on: Each object with a d-dimensional spinor index becomes an object with two spinor indices – one for the d_1-dimensional, one for the d_2-dimensional (s)pin group.

Case A has one further property. In the following, the goal will be to rewrite the parts of the split Lagrangian in the way they would look for a Lagrangian of d_1–dimensional space-time. That was already the rationale behind the Weyl rescaling, and we will continue with this program, below. But in such a d_1–dimensional Lagrangian, spinor conjugation will be a different operation from spinor conjugation in the $d_1 + d_2$–dimensional space-time – at least in case A, where $\overline{\Psi} = \Psi^\dagger\Gamma^0 = \Psi^\dagger\gamma^0\hat{\Gamma}^v$ is not the same as the d_1 dimensional conjugation $\Psi^\dagger\gamma^0$. Rewriting in a d_1–dimensional language, one should use d_1–dimensional spinor conjugation, which will be denoted by

$$\overline{\Psi}' := \Psi^\dagger\gamma^0 = \overline{\Psi}\;(\hat{\Gamma}^v)^{-1}. \tag{3.26}$$

Finally, a few words on conventions. I will, in the future, omit the explicit tensor product sign $\otimes$. Furthermore, the tensor factorization was defined using gamma matrices with upper, flat indices. For any γ or $\hat{\Gamma}$ occurring with different index types and/or positions, the transformation will have been made using e^a_m, $e'_\alpha{}^\mu$, $\bar{g}_{mn}$, $g'_{\mu\nu}$, $\eta'_{\alpha\beta}$ and $\bar{\eta}_{ab}$, thus retaining useful algebraic properties like $\{\gamma^\mu,\gamma^\nu\} = 2g'^{\mu\nu}$, $\{\gamma_\alpha,\gamma^\beta\} = 2\delta^\beta_\alpha$, and so on.

3.4 Supersymmetry variations of the vielbein fields

First, a look at the supersymmetry variations of the new vielbein components. The generic supersymmetry variation of a d-dimensional vielbein has the form

$$\delta_S E_M{}^A = \lambda\;(\bar{\epsilon}\,\Gamma^A\,\Psi_M)$$

where λ depends on the conventions chosen, and is not necessarily real or dimensionless. However, this supersymmetry variation will, in general, fail to preserve the special gauge that was chosen for the d-dimensional vielbein. In order to preserve the gauge choice, it is necessary that $\delta_S E_m{}^\alpha = 0$. This can be achieved by redefining the supersymmetry transformation in the following way. Let $\delta_{S'}$ be a supersymmetry variation that consists of the original transformation plus a special d–dimensional Lorentz rotation $\hat{\Lambda}$ that restores the gauge. The infinitesimal rotation $\hat{\Lambda}$ is defined by

$$\delta_{S'}E_m{}^\alpha \;=\; \delta_S E_m{}^\alpha + \hat{\Lambda}^\alpha{}_A E_m{}^A = \lambda\,\bar{\epsilon}\,\Gamma^\alpha\,\Psi_m + \hat{\Lambda}^\alpha{}_a E_m{}^a,$$

where the gauge $E_m{}^\alpha = 0$ has been used. Contracting with $e_b{}^m = E_b{}^m$ and using $\Psi_m e_b{}^m = \Psi_b$, we obtain

$$\hat{\Lambda}^\alpha{}_b = -\lambda\;\bar{\epsilon}\Gamma^\alpha\Psi_b. \tag{3.27}$$

As an infinitesimal Lorentz transformation, $\hat{\Lambda}_{AB}$ needs to be antisymmetric in A, B. Thus, so far, $\hat{\Lambda}_{\alpha b}$ and $\hat{\Lambda}_{b\alpha} = -\hat{\Lambda}_{\alpha b}$ are fixed; the other components are undetermined; I will choose $\hat{\Lambda}_{ab} = 0$ and give a suitable choice for $\hat{\Lambda}_{\alpha\beta}$ later on. With this redefinition, the modified supersymmetry transformations for the original vielbein are $\delta_{S'} E_m{}^\alpha = 0$ as well as

$$\begin{aligned} \delta_{S'} E_m{}^a &= \lambda\,(\bar\epsilon\,\Gamma^a \Psi_m) \\ \delta_{S'} E_\mu{}^\alpha &= \lambda\,(\bar\epsilon\,\Gamma^\alpha\,[\Psi_\mu - B_\mu{}^m \Psi_m]) + \Delta^s\,\hat\Lambda^\alpha{}_\beta e'_\mu{}^\beta = \lambda\,\Delta^s\,(\bar\epsilon\,\Gamma^\alpha e'_\mu{}^\beta \Psi_\beta) + \Delta^s\,\hat\Lambda^\alpha{}_\beta e'_\mu{}^\beta \\ \delta_{S'} E_\mu{}^a &= \lambda\,(\bar\epsilon\,[\Gamma^a \Psi_\mu + \Delta^s\,e'_\mu{}^\alpha \Gamma_\alpha \Psi^a]). \end{aligned} \tag{3.28}$$

Likewise, it is possible to write down the modified transformations for the inverse vielbein. Including the action of the compensating rotation $\hat\Lambda$, the result is the gauge-preservation $\delta_{S'} E_a{}^\mu = 0$ as well as

$$\begin{aligned} \delta_{S'} E_a{}^m &= -\lambda\,(\bar\epsilon\;[\Gamma^m + \Gamma^\mu B_\mu{}^m]\,\Psi_a) \\ \delta_{S'} E_\alpha{}^m &= -\lambda\,(\bar\epsilon\;[\Gamma^m \Psi_\alpha + \Gamma_\alpha \Psi^a e_a{}^m]) - \Delta^{-s}\hat\Lambda_\alpha{}^\beta e'_\beta{}^\mu B_\mu{}^m \\ \delta_{S'} E_\alpha{}^\mu &= -\lambda\,(\bar\epsilon\,\Gamma^\mu \Psi_\alpha) + \Delta^{-s}\hat\Lambda_\alpha{}^\beta e'_\beta{}^\mu. \end{aligned} \tag{3.29}$$

From this, one can calculate all the – modified – supersymmetry variations of the new vielbein fields. First of all, for the inner determinant Δ. As for any vielbein determinant, its variation is related to those of the vielbein by $\delta \ln \Delta = e_a{}^m \delta e_m{}^a$, so that

$$\delta_{S'} \Delta = \delta_{S'} \det e_m{}^a = \Delta\,\lambda\,(\bar\epsilon\,\Gamma^a \Psi_a). \tag{3.30}$$

For the other vielbein components, direct calculation then shows

$$\delta_{S'} e_m{}^a = \lambda\,(\bar\epsilon\,\Gamma^a \Psi_m) \tag{3.31}$$

$$\delta_{S'} B_\mu{}^m = \lambda\,\Delta^s\,(\bar\epsilon\,[\Gamma^a \Psi_\alpha + \Gamma_\alpha \Psi^a]) e'_\mu{}^\alpha e_a{}^m \tag{3.32}$$

$$\begin{aligned} \delta_{S'} e'_\mu{}^\alpha &= \lambda\,(\bar\epsilon\,\big[\Gamma^\alpha \Psi_\beta e'_\mu{}^\beta - s\,\Gamma^a \Psi_a e'_\mu{}^\alpha\big]) + \hat\Lambda^\alpha{}_\beta e'_\mu{}^\beta \\ &= \lambda\,(\bar\epsilon\,\big[\Gamma^\alpha \Psi^\beta - s\,\Gamma^\alpha \Gamma^\beta \Gamma^a \Psi_a\big]) e'_{\mu\beta} \\ &\quad + \lambda\,s\,(\bar\epsilon\,\Gamma^\alpha{}_\beta \Gamma^a \Psi_a) + \hat\Lambda^\alpha{}_\beta e'_\mu{}^\beta \end{aligned} \tag{3.33}$$

where, for the last reformulation, an $\eta^{\alpha\beta}$ has been inserted, and the Clifford property for the gamma matrices used. By its antisymmetry in α and β, the term containing $\Gamma^\alpha{}_\beta$ is an infinitesimal Lorentz transformation. It can be compensated for by choosing the as yet undetermined $\hat\Lambda^\alpha{}_\beta$ to be its negative, giving the vielbein variation the especially simple form

$$\delta_{S'} e'_\mu{}^\alpha = \lambda\,(\bar\epsilon\,\Gamma^\alpha\,[\Psi_\beta - s\,\Gamma_\beta \Gamma^a \Psi_a]) e'_\mu{}^\beta. \tag{3.34}$$

The same procedure can be repeated using the inverse vielbeins (although with $\hat\Lambda^\alpha{}_\beta$ already fixed). Alternatively, one can calculate their variation using $\delta_{S'}(e'_\alpha{}^\mu e'_\mu{}^\beta) = \delta_{S'}(e_a{}^m e_m{}^b) = 0$. The result is

$$\begin{aligned} \delta_{S'} e_a{}^m &= -\lambda e_b{}^m\,(\bar\epsilon\,\Gamma^b \Psi_a) = -\lambda\,(\bar\epsilon[\Gamma^m + \Gamma^\mu B_\mu{}^m]\Psi_a) \\ \delta_{S'} e'_\alpha{}^\mu &= -\lambda\,(\bar\epsilon\,\Gamma^\beta[\Psi_\alpha - s\Gamma_\alpha \Gamma^a \Psi_a]) e'_\beta{}^\mu. \end{aligned} \tag{3.35}$$

Next, use the fact that the gamma matrices will also be split into the matrices for d_1 and d_2 dimensions, as described in section 3.3. Let us start with case A, the split, $\Gamma^\alpha = \gamma^\alpha \hat{\Gamma}^v$ and $\Gamma^a = \hat{\Gamma}^a$. Taking care to lower flat space-time indices with $\eta'_{\alpha\beta}$ will lead to an extra factor ϵ_v, namely $\Gamma_\beta = \eta_{\alpha\beta}\Gamma^\beta = \zeta_1 \eta'_{\alpha\beta}\gamma^\beta\hat{\Gamma}^v = \epsilon_v \gamma_\beta \hat{\Gamma}^v$. With this, one can rewrite the variation(3.34) of ${e'_\mu}^\alpha$ as

$$\delta_{S'} {e'_\mu}^\alpha = \lambda \bar{\epsilon} \gamma^\alpha \hat{\Gamma}^v \left[\Psi_\beta - s\epsilon_v \gamma_\beta \hat{\Gamma}^v \hat{\Gamma}^a \Psi_a \right] {e'_\mu}^\beta. \tag{3.36}$$

Again, the goal is for the result of the dimensional split be, at least in part, an ordinary d_1-dimensional supergravity. However, as already stated at the very beginning of this chapter, the generic supersymmetry variation of a vielbein field has the simple form $\delta_S {e_\mu}^\alpha \sim \bar{\epsilon}\gamma^\alpha \Psi_\mu$ instead of that shown in (3.36). One can recover the proper form of the variation by redefining the new gravitino as

$$\Psi'_\mu \;\; = \;\; \Delta^r \left[\epsilon_v \Psi_\beta - s\gamma_\beta \hat{\Gamma}^v \hat{\Gamma}^a \Psi_a \right] {e'_\mu}^\beta. \tag{3.37}$$

(The rescaling with Δ^r is an optional extra, which does not change anything at this point as long as we define, at the same time, $\epsilon' = \Delta^{-r}\epsilon$. It will become important in the next section.) Then, there still is an out-of-place factor $\hat{\Gamma}^v$; however, that is absorbed once we introduce the proper d_1-dimensional Dirac conjugate $\overline{\epsilon}' = \epsilon'^{+}\gamma^0 = \epsilon_v \bar{\epsilon}' \hat{\Gamma}^v$, as described in section 3.3. All in all, these changes show the desired result – with the redefined fields, the variation is indeed

$$\delta_{S'} {e'_\mu}^\alpha = \lambda(\overline{\epsilon}' \gamma^\alpha \Psi'_\mu). \tag{3.38}$$

Case B is quite similar. The definition

$$\Psi'_\mu \;\; = \;\; \Delta^r [\Psi_\beta - s\gamma_\beta \gamma^v \hat{\Gamma}^a \Psi_a] {e'_\mu}^\beta \tag{3.39}$$

brings the supersymmetry variation (3.34) into the proper form where, again, the insertion of Δ^r will become important only in the next section and is compensated by a redefinition $\epsilon' = \Delta^{-r}\epsilon$ of the supersymmetry parameter.

3.5 Disentangling gravitino and matter fermions

In the preceding sections, we have defined the new gravitino Ψ'_μ. In addition, the d_1-dimensional theory features matter fermions that transform as spin-1/2 entities under the d_1-dimensional spin group. Their precursors, we already have met: those parts Ψ_a of the original gravitino Ψ_M that have no d_1-vector index. However, there remains one important check: With their current definition, are the degrees of freedom of the gravitino and fields properly block-diagonalized? In other words: Are there separate d_1-dimensional fermionic kinetic terms – containing two fermions and a space-time derivative – for matter fermions and gravitino, or are there mixed terms? It will turn out that, indeed, mixed terms are absent for a special choice of the rescaling Δ^r that has been introduced above.

With this question in mind, let us look at the d-dimensional Rarita-Schwinger term which, up to prefactors, is $E\,\overline{\Psi}_M\Gamma^{MNP}\partial_N\Psi_P$; however, in preparation for the split (for which we demand a certain index structure – lower, flat indices for Ψ, upper flat for the gammas), we can rewrite it as

$$\mathcal{L}_{RS} = E\,\overline{\Psi}_A\Gamma^{ABC}E_B{}^N\partial_N\Psi_C \tag{3.40}$$

(this holds up to a torsion term which leads to a contribution quartic in fermions, and as such shall not concern us). Of the resulting terms, keep only those with space-time derivatives ∂_μ (we will be more thorough in section 3.6, below).

Let us look at case A first. Substituting new for old gamma matrices using (3.23), introducing the new spinor conjugation $\overline{\Psi}$ defined above in (3.26), and using the vielbein split, the result is

$$\begin{aligned}\mathcal{L}_{RSs} \;=\; \Delta^s\,e'\Big[\; & \overline{\Psi}_\alpha\gamma^{\alpha\beta\gamma}\,e'_\beta{}^\nu\,\partial_\nu\Psi_\gamma + \epsilon_v\,\overline{\Psi}_\alpha\gamma^{\alpha\beta}\,\hat{\Gamma}^v\hat{\Gamma}^c e'_\beta{}^\nu\,\partial_\nu\Psi_c \\ & -\epsilon_v\overline{\Psi}_a\gamma^\beta\,\hat{\Gamma}^{ac}e'_\beta{}^\nu\,\partial_\nu\Psi_c + \epsilon_v\,\overline{\Psi}_a\gamma^{\beta\gamma}\,\hat{\Gamma}^v\hat{\Gamma}^a e'_\beta{}^\nu\,\partial_\nu\Psi_\gamma\Big].\end{aligned} \tag{3.41}$$

It is obvious that there are indeed undesirable mixed terms; however, it turns out that the gravitino redefinition (3.37) already does most of the trick. In the Rarita-Schwinger kinetic term for the d_1–dimensional redefined gravitino Ψ'_μ,

$$\mathcal{L}_{RSd_1} = e'\overline{\Psi}'_\mu\gamma^{\mu\nu\rho}\partial_\nu\Psi'_\rho, \tag{3.42}$$

insert both (3.37) and its new spinor adjoint

$$\overline{\Psi}'_\mu = \Delta^r[\epsilon_v\overline{\Psi}_\beta - s\overline{\Psi}_a\hat{\Gamma}^v\hat{\Gamma}^a\gamma_\beta]e'_\mu{}^\beta \tag{3.43}$$

(which, as there are only even numbers of gamma matrices involved, does not depend on the details of the spinor product – the two possible adjoints, Clifford reversion and Clifford conjugation, give the same results), and with a little help from gamma matrix algebra, the result is

$$\begin{aligned}\mathcal{L}_{RSd_1} \;=\; \Delta^{2r}e'\Big[\; & \overline{\Psi}_\beta\gamma^{\beta\delta\alpha}\,e'_\delta{}^\nu\partial_\nu\Psi_\alpha + \epsilon_v\,\overline{\Psi}_\beta\hat{\Gamma}^v\hat{\Gamma}^c\gamma^{\beta\delta}\,e'_\delta{}^\nu\partial_\nu\Psi_c \\ & -\epsilon_v\frac{(d_1-1)}{(d_1-2)}\,\overline{\Psi}_a\hat{\Gamma}^a\hat{\Gamma}^c\gamma^\delta\,e'_\delta{}^\nu\partial_\nu\Psi_c + \epsilon_v\,\overline{\Psi}_a\hat{\Gamma}^v\hat{\Gamma}^a\gamma^{\delta\alpha}\,e'_\delta{}^\nu\partial_\nu\Psi_\alpha\Big].\end{aligned}$$

All that is missing to complete the picture is the choice of $r = s/2$, and one can rewrite the truncated Rarita-Schwinger term as

$$\mathcal{L}_{RSs} \;=\; e'\overline{\Psi}'_\mu\gamma^{\mu\nu\rho}\partial_\nu\Psi'_\rho + \Delta^s\,e'\,\epsilon_v\,\overline{\Psi}_a\left[\eta^{ac} + \frac{1}{(d_1-2)}\hat{\Gamma}^a\hat{\Gamma}^c\right]\gamma^\beta\,e'_\beta{}^\nu\,\partial_\nu\Psi_c.$$

Some further simplifications are possible: the extra powers of Δ can be absorbed by introducing rescaled fields $\tilde{\chi}_a := \Delta^{s/2}\Psi_a$, and, furthermore, one can introduce a field redefinition

$$\chi_a := \tilde{\chi}_a + \bar{c}\hat{\Gamma}_a\hat{\Gamma}^c\tilde{\chi}_c \qquad \Rightarrow \qquad \tilde{\chi}_a = \chi_a - \frac{\bar{c}}{1+\bar{c}d_2}\hat{\Gamma}_a\hat{\Gamma}^c\chi_c, \tag{3.44}$$

which satisfies

$$\overline{\chi}^a\,\chi_a = \overline{\tilde{\chi}}_a\left[\eta^{ab} + \hat{\Gamma}^a\hat{\Gamma}^b(2c + d_2c^2)\right]\tilde{\chi}_b, \tag{3.45}$$

to bring the kinetic term of the χ_a into a more convenient form. However, in the hunt for hidden symmetries, there are usually further redefinitions of the χ_a, and as these will determine which of the possible forms is most suitable, the choice of $\bar{c}$ will be postponed.

For case B, both the recipe and the result are almost exactly the same. Inserting the vielbein split and the case B split (3.25) of the gamma matrices, one obtains an expression with mixed terms; using the redefined Ψ'_μ and choosing $r = s/2$, gamma matrix algebra leads to the result

$$\mathcal{L}_{RSs} \;=\; e'\overline{\Psi}'_\mu\gamma^{\mu\nu\rho}\partial_\nu\Psi'_\rho + \Delta^s\, e'\,\epsilon_v\overline{\Psi}_a\left[\eta^{ac} + \frac{1}{(d_1-2)}\hat{\Gamma}^a\hat{\Gamma}^c\right]\gamma^\beta\, e'_\beta{}^\nu\,\partial_\nu\Psi_c,$$

which, apart from the subtle difference between $\overline{\Psi}$ and $\overline{\Psi}$, is exactly the same as for case A.

3.6 Splitting the gravitino's kinetic term

In the preceding section, we have already taken a stab at splitting the original gravitino's kinetic term. However, our attention was restricted to selected contributions only, in order to see how the redefinition and proper rescaling of the d_1-dimensional gravitino disentangles its degrees of freedom from those of the spin-1/2 fermions. In this section, a more complete treatment will be presented, keeping all terms that are quadratic in the fermions – not only all parts of the new gravitino kinetic term and the kinetic term of the new spin-1/2 fermions, but also the mixed term that contains one Ψ and one $\tilde{\chi}$. Quartic and higher fermionic terms (arising from torsion) will be neglected throughout.

Let us restrict ourselves to case A. The starting point is, once more, the original Rarita-Schwinger term, rewritten throughout with flat indices,

$$E(\overline{\Psi}_M\Gamma^{MNP}D_N\Psi_P) = E(\overline{\Psi}_A\Gamma^{ABC}E_B{}^N(D_N\Psi)_C), \tag{3.46}$$

which holds true neglecting a torsion term. In the following, rescale Ψ_a to $\tilde{\chi}_a$, and substitute Ψ'_μ for Ψ_α. In the course of these replacement operations, there will be terms in which the derivative D_N acts not on fermionic fields, but on factors Δ or

${e'}_\alpha{}^\mu$ introduced by the aforementioned redefinitions. Let us deal with the latter terms first. At the same time, it is convenient to split the covariant derivative by defining as a separate entity

$$\bar{D}_N \Psi_C = (D_N \Psi)_C - \omega_{NC}{}^D \Psi_D, \tag{3.47}$$

that part of the derivative covariant with respect to spinor indices, only. The result of this procedure is

$$E\Delta^{-s/2}\Big\{\epsilon_v(\overline{\Psi}_A \Gamma^{AN\gamma} e'^{\,\mu}_\gamma (D_N \Psi'_\mu)) + s(\overline{\Psi}_A \Gamma^{AN\gamma} \bar{D}_N(\Gamma_\gamma \Gamma^c \tilde{\chi}_c) + (\overline{\Psi}_A \Gamma^{ANc}(\bar{D}_N \tilde{\chi}_c))$$
$$+ \epsilon_v s(\overline{\Psi}_A \Gamma^{AN\gamma} e'^{\,\mu}_\gamma \Psi'_\mu)(\partial_N \ln \Delta) - \epsilon_v(\overline{\Psi}_A \Gamma^{ANP} \Psi'_\mu) e'^{\,\mu}_\delta (D_{[N} E_{P]}{}^\delta)$$
$$+ (\overline{\Psi}_A \Gamma^{ANC}[s\omega_{NC}{}^\delta \Gamma_\delta \Gamma^d + \omega_{NC}{}^d]\tilde{\chi}_d)\Big\}. \tag{3.48}$$

(In fact, if one discards the factors ϵ_v, this decomposition is valid for case B, as well.) On Ψ'_μ, D acts just as $\bar{D}$ does. There, it suffices to examine how the derivative $\bar{D}$ can be split. The result is

$$\begin{aligned}(\bar{D}_\nu - B_\nu{}^n \bar{D}_n) \;&=\; \mathcal{D}_\nu + \frac{1}{4}\omega'_{\nu\delta\varepsilon}\gamma^{\delta\epsilon} + \frac{s}{2}(\mathcal{D}_\rho \ln\Delta)\gamma_\nu{}^\rho + \frac{s\epsilon_v}{2}\Delta^s(\partial_n \ln\Delta)\hat{\Gamma}^v\hat{\Gamma}^n\gamma_\nu \\ &\quad -\frac{1}{4}{e'}_\nu{}^\gamma(2\epsilon_v\Delta^s\Omega'_{e(\gamma\delta)} - \Delta^{-s}\Omega'_{\delta\gamma e})\hat{\Gamma}^v\hat{\Gamma}^e\gamma^\delta - \frac{1}{4}{e'}_\nu{}^\gamma\Omega'_{\gamma de}\hat{\Gamma}^{de},\end{aligned} \tag{3.49}$$

where $\mathcal{D}_\mu := \partial_\mu - B_\mu{}^m \partial_m$, and

$$\begin{aligned}\bar{D}_n \;&=\; \partial_n + \frac{1}{8}e_n{}^c(\epsilon_v\Delta^{-2s}\Omega'_{\delta\varepsilon c} - 2\Omega'_{c[\delta\varepsilon]})\gamma^{\delta\varepsilon} \\ &\quad +\frac{1}{2}\Delta^{-s}\hat{\Gamma}^v\hat{\Gamma}^b e_n{}^c\gamma^\alpha\Omega'_{\alpha(bc)} + \frac{1}{8}e_n{}^c(\Omega'_{dec} - 2\Omega'_{c[de]})\hat{\Gamma}^{de}.\end{aligned} \tag{3.50}$$

Let us start by rewriting those terms in (3.48) of the form $\overline{\Psi}' \cdots \Psi'$. Part of the work has already been done in section 3.5; the result is

$$\begin{aligned}&(\overline{\Psi}'_\mu\gamma^{\mu\nu\rho}[D_\nu - B_\nu{}^m D_m]\Psi'_\rho) - \epsilon_v\Delta^s(\overline{\Psi}'_\mu\gamma^{\mu\nu}\hat{\Gamma}^v\hat{\Gamma}^m D_m\Psi'_\nu) \\ &\quad = \; (\overline{\Psi}'_\mu\gamma^{\mu\nu\rho}\left[\mathcal{D}_\nu + \tfrac{1}{4}\omega'_{\nu\alpha\beta}\gamma^{\alpha\beta}\right]\Psi'_\rho) \\ &\qquad -\frac{1}{4}(\overline{\Psi}'_\mu\gamma^{\mu\gamma\rho}\hat{\Gamma}^{de}\Psi'_\rho)\Omega'_{\gamma de} + \frac{\Delta^{-s}}{4}(\overline{\Psi}'_\mu\hat{\Gamma}^v\hat{\Gamma}^e\Psi'_\nu)e'^{\mu\delta}e'^{\nu\varepsilon}\Omega'_{\delta\varepsilon e} \\ &\qquad -\epsilon_v\Delta^s(\overline{\Psi}'_\mu\gamma^{\mu\nu}\hat{\Gamma}^v\hat{\Gamma}^n\left[\partial_n + \tfrac{1}{4}\gamma^{\delta\varepsilon}e'^{\,\rho}_\delta(\partial_n e'_{\rho\varepsilon})\right]\Psi'_\nu) \\ &\qquad -\frac{\epsilon_v\Delta^s}{4}(\overline{\Psi}'_\mu\gamma^{\mu\nu}\hat{\Gamma}^v\hat{\Gamma}^{cde}\Psi'_\nu)e_c{}^m e_d{}^n(\partial_m e_{ne}) + (\overline{\Psi}'_\mu\gamma^\mu\Psi'^\sigma)(\partial_n B_\sigma{}^n) \\ &\qquad +\frac{\Delta^{-s}}{8}(\overline{\Psi}'_\mu\gamma^{\mu\nu\delta\varepsilon}\hat{\Gamma}^v\hat{\Gamma}^c\Psi'_\nu)\Omega'_{\delta\varepsilon c}.\end{aligned} \tag{3.51}$$

In the special case that is dimensional reduction, where all dependence on the internal coordinates is dropped, this expression simplifies to just five differently structured terms

$$(\overline{\Psi}'_\mu \gamma^{\mu\nu\rho} \left[\partial_\nu + \tfrac{1}{4}\omega'_{\nu\alpha\beta}\gamma^{\alpha\beta}\right] \Psi'_\rho) + \frac{\Delta^{-s}}{8}(\overline{\Psi}'_\mu \gamma^{\mu\nu\delta\varepsilon}\hat{\Gamma}^v\hat{\Gamma}^c\Psi'_\nu)\Omega'_{\delta\varepsilon c}$$
$$-\frac{1}{4}(\overline{\Psi}'_\mu \gamma^{\mu\gamma\rho}\hat{\Gamma}^{de}\Psi'_\rho)\Omega'_{\gamma de} + \frac{\Delta^{-s}}{4}(\overline{\Psi}'_\mu \hat{\Gamma}^v\hat{\Gamma}^e\Psi'_\nu)e'^{\mu\delta}e'^{\nu\varepsilon}\Omega'_{\delta\varepsilon e}. \tag{3.52}$$

In the general case, the interpretation of the terms in this expression is not straightforward. However, in the special case $d_1 = 3$ (which, conveniently, happens to be the case of interest later on), matters simplify considerably, as follows. For a three-dimensional Clifford algebra with signature $(-,+,+)$, define ϵ_3 by $\gamma^1\gamma^2\gamma^3 = \epsilon_3\mathbb{1}$. Then, a gamma matrix relation[5] $\varepsilon^{\alpha\beta\gamma} = -\epsilon_3\gamma^{\alpha\beta\gamma}$ holds, with the help of which the last term in (3.52) can be rewritten. Namely, introducing the Hodge dual of the anholonomy coefficient $\Omega'_{\delta\varepsilon e}$,

$$\Omega'_{\alpha e} = \frac{1}{2}\varepsilon_\alpha{}^{\delta\varepsilon}\Omega'_{\delta\varepsilon e} \quad \Rightarrow \quad \Omega'_{\delta\varepsilon e} = (-)^q \varepsilon_{\delta\varepsilon}{}^\alpha \Omega'_{\alpha e}, \tag{3.53}$$

where q is the number of negative eigenvalues of the flat metric, η', one can rewrite $\Omega'_{\delta\varepsilon e}$ as

$$\Omega'_{\delta\varepsilon e} = -(-)^q \epsilon_3 \gamma_{\delta\varepsilon}{}^\alpha \Omega'_{\alpha e} = (\epsilon_3)^{-1}\gamma_{\delta\varepsilon}{}^\alpha\Omega'_{\alpha e}. \tag{3.54}$$

The redefinition makes it possible to rewrite the dimensionally reduced expression to obtain a unified (space-time-)gamma matrix structure,

$$(\overline{\Psi}'_\mu \gamma^{\mu\nu\rho}\left[\partial_\nu + \tfrac{1}{4}\omega'_{\nu\alpha\beta}\gamma^{\alpha\beta} - \tfrac{1}{4}e'_\nu{}^\alpha(\hat{\Gamma}^{de}\Omega'_{\alpha de} + (\epsilon_3)^{-1}\,\Delta\,\hat{\Gamma}^v\hat{\Gamma}^e\Omega'_{\alpha e})\right]\Psi'_\rho). \tag{3.55}$$

This can be read as a modified Rarita-Schwinger term with a covariant derivative that includes not only the Lorentz connection $\omega'_{\nu\alpha\beta}$, but also (parts of) a connection acting on the internal spinor indices. We will see later on that the introduction of this new form of the covariant derivative is consistent with the rewriting of another part of the supergravity that generically contains a covariant derivative, namely the variation of the new gravitino, $\delta\Psi'_\mu = D\epsilon + \ldots$, which will be calculated in section 3.7. In the rewriting of theories with hidden symmetries, the new part of the connection is identified as the connection associated with the (enlarged) local symmetry H.

Next, it is time to look at the mixed terms occurring in (3.48), i.e. the terms that, after introducing the new gravitino, contain exactly one Ψ' and one matter fermion $\tilde{\chi}$. Performing the split, substituting, once more, the gamma matrices (3.23), the new gravitino definition (3.37) and the elements of $\bar{D}$ given above, one obtains a set of about four dozen terms; however, judicious use of the Clifford algebra relations makes for substantial and gratifying simplification. Let us choose to move all Ψ' to the right of the respective spinor products (this is always possible without knowing the exact

[5] The convention used for flat epsilon symbols is $\varepsilon_{123} = +1$.

form of the spinor product's symmetry – the fact that the original spinor product was such that the original Rarita-Schwinger Lagrangian was non-vanishing provides all the information needed). In one case, this motivates performing a partial integration with respect to ∂_n. In the end, the terms recombine to give

$$e'\Big\{ \; 2\Delta^s(\bar{\tilde{\chi}}_a[s\hat{\Gamma}^a\hat{\Gamma}^n - e^{an}]\gamma^\rho[\partial_n + \tfrac{1}{2}(s+1)(\partial_n \ln\Delta) + \tfrac{1}{4}(e'^{\ \nu}_\delta \partial_n e'_{\nu\varepsilon})\gamma^{\delta\varepsilon}]\Psi'_\rho)$$
$$+ \frac{\Delta^{-s}\epsilon_v}{4}(\bar{\tilde{\chi}}_a[s\hat{\Gamma}^a\hat{\Gamma}^c - \bar{\eta}^{ac}]\gamma^\rho\gamma^{\delta\varepsilon}\Psi'_\rho)\Omega'_{\delta\varepsilon c}$$
$$+ (\bar{\tilde{\chi}}^b\hat{\Gamma}^v\hat{\Gamma}^c\gamma^\rho\gamma^\alpha\Psi'_\rho)[s\bar{\eta}_{bc}\Omega'^{\ \ d}_{\alpha d} - \Omega'_{\alpha(bc)}]$$
$$+ \frac{\Delta^s}{2}(\bar{\tilde{\chi}}_a[(s\hat{\Gamma}^a\hat{\Gamma}^b - \bar{\eta}^{ab})\hat{\Gamma}^{cd}$$
$$+ 2(\bar{\eta}^{ab}\bar{\eta}^{cd} + \hat{\Gamma}^{b(c}\bar{\eta}^{d)a} - s\hat{\Gamma}^a\hat{\Gamma}^{(c}\bar{\eta}^{d)b}]\gamma^\rho\Psi'_\rho)e_b{}^m e_c{}^n(\partial_m e_{nd})\Big\}, \quad (3.56)$$

where a boundary term $-\partial_n[e'\Delta^s(\bar{\tilde{\chi}}_a[s\hat{\Gamma}^a\hat{\Gamma}^n - e^{an}]\gamma^\rho\Psi'_\rho)]$ has been neglected. Note that there are certain structural similarities with (3.51)– for instance, the internal derivative ∂_n occurring in the company of $1/4\cdot(e'^{\ \nu}_\delta\partial_n e'_{\nu\varepsilon})\gamma^{\delta\varepsilon}$.

Evidently, restriction to the special case $d_1 = 3$, corresponding to $s = -1$, again brings about some further simplifications. One can dualize $\Omega_{\delta\epsilon c}$ as defined in (3.53), and use the gamma matrix identity in three dimensions, $2\epsilon_3\gamma^\alpha = \varepsilon^{\alpha\beta\delta}\gamma_{\beta\delta}$ to absorb the epsilon symbol. The result is

$$e'\Big\{ \; -2\Delta^{-1}(\bar{\tilde{\chi}}_a[\hat{\Gamma}^a\hat{\Gamma}^n + e^{an}]\gamma^\rho[\partial_n + \tfrac{1}{4}(e'^{\ \nu}_\delta\partial_n e'_{\nu\varepsilon})\gamma^{\delta\varepsilon}]\Psi'_\rho)$$
$$+ \frac{\Delta\epsilon_v(\epsilon_3)^{-1}}{2}(\bar{\tilde{\chi}}_a[\hat{\Gamma}^a\hat{\Gamma}^c + \bar{\eta}^{ac}]\gamma^\rho\gamma^\alpha\Psi'_\rho)\Omega'_{\alpha c} - (\bar{\tilde{\chi}}^b\hat{\Gamma}^v\hat{\Gamma}^c\gamma^\rho\gamma^\alpha\Psi'_\rho)[\bar{\eta}_{bc}\Omega'^{\ \ d}_{\alpha d} + \Omega'_{\alpha(bc)}]$$
$$+ \frac{\Delta^{-1}}{2}(\bar{\tilde{\chi}}_a[-(\hat{\Gamma}^a\hat{\Gamma}^b + \bar{\eta}^{ab})\hat{\Gamma}^{cd}$$
$$+ 2(\bar{\eta}^{ab}\bar{\eta}^{cd} + \hat{\Gamma}^{b(c}\bar{\eta}^{d)a} + \hat{\Gamma}^a\hat{\Gamma}^{(c}\bar{\eta}^{d)b}]\gamma^\rho\Psi'_\rho)e_b{}^m e_c{}^n(\partial_m e_{nd})\Big\}. \quad (3.57)$$

In the case of dimensional reduction, dropping all dependence on the internal coordinates reveals that, thanks to the dualization, the surviving terms all have the same simple (space-time-)gamma matrix structure, namely $(\bar{\tilde{\chi}}\gamma^\rho\gamma^\alpha\Psi_\rho)$. Sure enough, that is the proper structure for a Noether term in a three-dimensional coset model (compare [93], the more general [42] or, closer to hand, this thesis' very own example (2.36) in chapter 2).

Finally for the terms in (3.48) that contain two matter fermions $\tilde{\chi}$. Performing the required steps that implement the dimensional split (and with which the reader is by now thoroughly familiar) those terms turn out to be

$$
\begin{aligned}
e'\Big\{ &- \epsilon_v(\overline{\tilde{\chi}}_a[s\hat{\Gamma}^a\hat{\Gamma}^c - \bar{\eta}^{ac}]\gamma^\nu[\mathcal{D}_\nu + \tfrac{1}{4}\gamma^{\delta\varepsilon}\omega'_{\nu\delta\varepsilon}]\tilde{\chi}_c) \\
&+ \Delta^s(\overline{\tilde{\chi}}_a\hat{\Gamma}^v\big\{\hat{\Gamma}^{anc} - (2s-1)[e^{an}\hat{\Gamma}^c - s\hat{\Gamma}^a e^{cn}] + (2s-1)(s+1)\hat{\Gamma}^a\hat{\Gamma}^{nc}\big\} \\
&\qquad\qquad \cdot [\partial_n + \tfrac{1}{4}(e'^{\ \nu}_{\delta}\partial_n e'_{\nu\varepsilon})\gamma^{\delta\varepsilon}]\tilde{\chi}_c) \\
&+ \frac{\epsilon_v}{8}\Delta^{-s}\Omega'_{\delta\varepsilon d}(\overline{\tilde{\chi}}_a\hat{\Gamma}^v[(2s+1)s\hat{\Gamma}^{acd} + (2s-1)(s+1)\bar{\eta}^{ac}\hat{\Gamma}^d \\
&\qquad\qquad - (2s-1)s(\hat{\Gamma}^a\bar{\eta}^{cd} + \hat{\Gamma}^c\bar{\eta}^{ad})]\gamma^{\delta\varepsilon}\tilde{\chi}_c) \\
&+ \frac{\epsilon_v}{4}\Omega'_{\alpha[bd]}(\overline{\tilde{\chi}}_a[-s\hat{\Gamma}^{abcd} + (s-1)\hat{\Gamma}^{bd}\bar{\eta}^{ac} + 4s\hat{\Gamma}^{ab}\bar{\eta}^{cd} + 2(s-2)\bar{\eta}^{ab}\bar{\eta}^{cd}]\gamma^\alpha\tilde{\chi}_c) \\
&- \frac{1}{4}\Delta^s(e_d{}^m e_e{}^n\partial_m e_{nf})(\overline{\tilde{\chi}}_a\hat{\Gamma}^v[\ (2s+1)s\hat{\Gamma}^{acdef} + (s+1)(2s-1)\bar{\eta}^{ac}\hat{\Gamma}^{def} \\
&\qquad\qquad - 2s(2s+3)\bar{\eta}^{cf}\hat{\Gamma}^{ade} + 4(2s^2+s-2)\bar{\eta}^{af}\bar{\eta}^{c[d}\hat{\Gamma}^{e]} \\
&\qquad\qquad - 2s(2s-1)(\hat{\Gamma}^{aef}\bar{\eta}^{cd} - \hat{\Gamma}^{adf}\bar{\eta}^{ce}) \\
&\qquad\qquad + 2(2s^2-3s+2)\bar{\eta}^{ad}\bar{\eta}^{ce}\hat{\Gamma}^f + 16s\hat{\Gamma}^a\bar{\eta}^{c[d}\bar{\eta}^{e]f}]\tilde{\chi}_c)\Big\}.
\end{aligned}
$$

Yet again, the case $d_1 = 3 \Rightarrow s = -1$ proves to be special. Some terms vanish outright, and it is, once more, possible to dualize $\Omega'_{\delta\epsilon d}$ to $\Omega'_{\alpha d}$ as in (3.53) in order to obtain the somewhat simpler expression

$$
\begin{aligned}
e'\Big\{ &\ \epsilon_v(\overline{\tilde{\chi}}_a[\hat{\Gamma}^a\hat{\Gamma}^c + \bar{\eta}^{ac}]\gamma^\nu[\mathcal{D}_\nu + \tfrac{1}{4}\gamma^{\delta\varepsilon}\omega'_{\nu\delta\varepsilon}]\tilde{\chi}_c) \\
&+ \Delta^{-1}(\overline{\tilde{\chi}}_a\hat{\Gamma}^v\big\{\hat{\Gamma}^{anc} + 3[e^{an}\hat{\Gamma}^c + \hat{\Gamma}^a e^{cn}]\big\}[\partial_n + \tfrac{1}{4}(e'^{\ \nu}_{\delta}\partial_n e'_{\nu\varepsilon})\gamma^{\delta\varepsilon}]\tilde{\chi}_c) \\
&- \frac{(\epsilon_3)^{-1}\epsilon_v}{4}\Delta\Omega'_{\alpha d}(\overline{\tilde{\chi}}_a\hat{\Gamma}^v[\hat{\Gamma}^{acd} - 3(\hat{\Gamma}^a\bar{\eta}^{cd} + \hat{\Gamma}^c\bar{\eta}^{ad})]\gamma^\alpha\tilde{\chi}_c) \\
&+ \frac{\epsilon_v}{4}\Omega'_{\alpha[bd]}(\overline{\tilde{\chi}}_a[\hat{\Gamma}^{abcd} - 2\hat{\Gamma}^{bd}\bar{\eta}^{ac} - 4\hat{\Gamma}^{ab}\bar{\eta}^{cd} - 6\bar{\eta}^{ab}\bar{\eta}^{cd}]\gamma^\alpha\tilde{\chi}_c) \\
&- \frac{1}{4}\Delta^{-1}(e_d{}^m e_e{}^n\partial_m e_{nf})(\overline{\tilde{\chi}}_a\hat{\Gamma}^v[\ \hat{\Gamma}^{acdef} + 2\bar{\eta}^{cf}\hat{\Gamma}^{ade} - 4\bar{\eta}^{af}\bar{\eta}^{c[d}\hat{\Gamma}^{e]} \\
&\qquad\qquad - 6(\hat{\Gamma}^{aef}\bar{\eta}^{cd} - \hat{\Gamma}^{adf}\bar{\eta}^{ce}) \\
&\qquad\qquad + 14\bar{\eta}^{ad}\bar{\eta}^{ce}\hat{\Gamma}^f - 16\hat{\Gamma}^a\bar{\eta}^{c[d}\bar{\eta}^{e]f}]\tilde{\chi}_c)\Big\}. \qquad (3.58)
\end{aligned}
$$

After dimensional reduction, the result is again what looks like a standard kinetic term (this time for spin 1/2) with a modified covariant derivative.

3.7 Supersymmetry variations of the new gravitino field

In this section, we will take a look at the supersymmetry variations of the redefined gravitino field, Ψ'_μ. Given that the variations of Ψ_M are dependent on the details of the d-dimensional theory, a suitably generic analysis such as in the case of the vielbein variations is beyond the scope of this chapter. In the following, I will at least present some results dealing with those contributions to the variation that are suitably generic and thus can be accounted for in a more general setting: first, the contribution from the Lorentz-covariant derivative $D_M\epsilon$, later, on p. 49f., an example for the contribution of a p-form field to the gravitino's supersymmetry variation, choosing a standard Maxwell field, whose field-strength is a two-form, such as will be needed for the five-dimensional supergravity, later.

For a start, there are some preparations that are independent of the exact form of the original gravitino variation $\delta_S\Psi_M$, namely establishing the relationship between the variation $\delta_S\Psi'_\mu$ of the new d_1-dimensional gravitino field and the components $\delta_S\Psi_\mu$ and $\delta_S\Psi_m$ of the d-dimensional variations $\delta_S\Psi_M$. Once more, only case A will be treated, where the starting point is the definition (3.37) of Ψ'_μ.

In order to use the information about the variation of the original gravitino Ψ_M which follows from the higher-dimensional theory, all flat-component fields Ψ_α and Ψ_a will have to be rewritten in terms of curved components, using the equations (3.7) and (3.8) from section 3.1 while, in the end, it will be necessary to rewrite everything in terms of the redefined fields, such as Ψ'_μ. In this there-and-back-again procedure, it proves advantageous to treat the scaling factor Δ^r separately, namely to start the analysis by computing

$$\delta\Psi'_\mu = r(\delta\ln\Delta)\Psi'_\mu + \Delta^r\delta(\Delta^{-r}\Psi'_\mu),$$

where the rewriting in terms of curved components is now only necessary for $\Delta^{-r}\Psi'_\mu$, an expression without an overall scaling factor. Next, the splitting procedure: Taking care to insert, wherever possible, the redefined Ψ'_μ and the rescaled field $\tilde{\chi}_a = \Delta^r\Psi_a$, and inserting the variations of the vielbein field defined above in eqq. (3.31) to (3.33), the intermediate result is

$$\begin{aligned}\delta\Psi'_\mu \;\;=\;\; & \Delta^{-r}\epsilon_v\left[\delta\Psi_\mu - (B_\mu{}^m + \epsilon_v s\Delta^s\gamma_\beta e'_\mu{}^\beta\hat{\Gamma}^v\hat{\Gamma}^a e_a{}^m)\delta\Psi_m\right] \\ & +\lambda\Big\{ - s(\overline{\epsilon}'\gamma^\beta\Psi'_\mu)\gamma_\beta\hat{\Gamma}^v\hat{\Gamma}^a\tilde{\chi}_a - \epsilon_v\tilde{\chi}_a(\overline{\epsilon}'\gamma^\beta e'_\mu{}^\beta[\eta^{ab} - s\hat{\Gamma}^a\hat{\Gamma}^b]\tilde{\chi}_b) \\ & \quad + (\overline{\epsilon}'\hat{\Gamma}^v\hat{\Gamma}^a\tilde{\chi}_a)\left[-s/2\;\Psi'_\mu - s^2\gamma_\beta e'_\mu{}^\beta\hat{\Gamma}^v\hat{\Gamma}^b\tilde{\chi}_b\right] \\ & \quad - (\overline{\epsilon}'\hat{\Gamma}^v\hat{\Gamma}^a\Psi'_\mu)\tilde{\chi}_a + s\gamma_\beta e'_\mu{}^\beta\hat{\Gamma}^v\hat{\Gamma}^a\tilde{\chi}_b(\overline{\epsilon}'\hat{\Gamma}^v\hat{\Gamma}^b\tilde{\chi}_a)\Big\}. \end{aligned} \tag{3.59}$$

On this basis, let us now consider some specific contributions to $\delta\Psi_\mu$ and $\delta\Psi_m$.

Contributions from the Lorentz covariant derivative

However the details of the gravitino's supersymmetry variations may vary between the different supergravity models, one contribution is universal, namely a covariant derivative of the supersymmetry parameter, $D_M\epsilon$. Now, for a look at this term's contribution to the dimensionally reduced gravitino's variation, via the $\delta\Psi_\mu$ and $\delta\Psi_m$. Written out, and taking our cue from eq. (3.59), that contribution is

$$\delta\Psi'_\mu = \Delta^{-r}\epsilon_v\left[D_\mu\epsilon - (B_\mu{}^m + \epsilon_v s\Delta^s\gamma_\beta e'_\mu{}^\beta\hat\Gamma^v\hat\Gamma^a e_a{}^m)D_m\epsilon\right] + \text{ rest} .$$

As the ϵ has no non-spinorial Lorentz indices, D_μ and D_m are just the expressions $\bar D_\mu$ and $\bar D_m$ that were given in eq. (3.49) and (3.50), respectively – that is, leaving out higher fermionic terms, namely the contorsion contributions to the covariant derivative, as we shall do in the following. After suitable applications of the Clifford relation, and resolving some of the anholonomy coefficients, the result is

$$\begin{aligned}\delta\Psi'_\mu \;=\; \epsilon_v\Bigg\{ & \mathcal{D}_\mu + \frac{1}{4}\gamma^{\alpha\beta}\omega'_{\mu\alpha\beta} + \frac{s}{2}\gamma_\mu\gamma^\nu(\partial_m B_\nu{}^m) \\ & + \frac{1}{4}\hat\Gamma^{ab}e'_\mu{}^\delta\Omega'_{a\delta b} - \frac{s}{8}\Delta^{-s}\hat\Gamma^v\hat\Gamma^c\gamma_\mu{}^{\alpha\beta}\Omega'_{\alpha\beta c} \\ & - \epsilon_v s\Delta^s\gamma_\mu\hat\Gamma^v\hat\Gamma^m\left[\partial_m + \frac{(s-1)}{2}(\partial_m\ln\Delta) + \frac{1}{4}\hat\Gamma^{ab}\omega_{mab}\right] \\ & + \frac{\epsilon_v}{4}\Delta^s\hat\Gamma^v\hat\Gamma^m\left[(1-s)\gamma^\alpha\delta^\nu_\mu - s\,\gamma_\mu{}^{\nu\alpha}\right](\partial_m e'_{\nu\alpha}) \\ & + \frac{(1+s)}{4}\hat\Gamma^v\hat\Gamma^b e'_\mu{}^\delta\gamma^\alpha\left[\epsilon_v\Delta^s\Omega'_{\alpha b\delta} - \Delta^{-s}\Omega'_{\delta\alpha b}\right]\Bigg\}\epsilon'. \end{aligned} \tag{3.60}$$

In the case of dimensional reduction, with no dependence of the fields on the internal coordinates, this simplifies to

$$\begin{aligned}\delta\Psi'_\mu \;=\; \epsilon_v\Bigg\{ & \partial_\mu + \frac{1}{4}\gamma^{\alpha\beta}\omega'_{\mu\alpha\beta} + \frac{1}{4}\hat\Gamma^{ab}e'_\mu{}^\delta\Omega'_{a\delta b} - \frac{s}{8}\Delta^{-s}\hat\Gamma^v\hat\Gamma^c\gamma_\mu{}^{\alpha\beta}\Omega'_{\alpha\beta c} \\ & + \frac{(1+s)}{4}\Delta^{-s}\hat\Gamma^v\hat\Gamma_m\gamma^\nu(\partial_\mu B_\nu{}^m - \partial_\nu B_\mu{}^m)\Bigg\}\epsilon'. \end{aligned} \tag{3.61}$$

The first two terms constitute the Lorentz covariant space-time derivative D'_μ acting on ϵ'. The other terms are less obvious to interpret. However, it can be seen that once again, $d_1 = 3$ is an especially simple case: The last term vanishes because of $s = -1$, and the other terms's action on space-time spinor indices is that of the identity, because of the proportionality $\gamma_\mu{}^{\alpha\beta} \sim \varepsilon_\mu{}^{\alpha\beta}$. In fact, the presence of the $\varepsilon_\mu{}^{\alpha\beta}$ is downright suggestive of the dualization $\Omega'_{\alpha\beta c} \mapsto \Omega'_{\delta c}$ introduced in eq. (3.53), above. With these substitutions, we have

$$\delta\Psi'_\mu \;=\; \epsilon_v\left\{\partial_\mu + \frac{1}{4}\gamma^{\alpha\beta}\omega'_{\mu\alpha\beta} - \frac{1}{4}e'_\mu{}^\delta\left[\hat{\Gamma}^{ab}\Omega'_{\delta ab} + \Delta(\epsilon_3)^{\text{-}1}\hat{\Gamma}^v\hat{\Gamma}^c\Omega'_{\delta c}\right]\right\}\epsilon' \quad (3.62)$$

This is exactly the enhanced covariant derivative that we had encountered in the reduction of the Rarita-Schwinger term (3.55), involving the three-dimensional spin connection $\omega'_{\mu\alpha\beta}$ and new, additional connection terms.

Contributions of p-form fields: an example

In the simpler supergravity theories, such as $\mathcal{N}=1$ in $d=3,4$, the Lorentz-covariant derivative $D\epsilon$ of the supersymmetry parameter is the only contribution to the graviton's supersymmetry variation. In more complicated theories, that variation can usually be rewritten as the action of a so-called supercovariant derivative, $\hat{D}\epsilon$, which involves not only the spin connection but certain contributions of the p-form field that is part of the bosonic section of the supergravity multiplet. For such a contribution, a generic form is [4]

$$\delta_s\Psi = \hat{D}\epsilon = D\epsilon + b_1(\Gamma_{A_1\cdots A_m C} + b_2\eta_{A_m C}\Gamma_{A_1\cdots A_{m-1}})F^{A_1\cdots A_m}E^C\epsilon \quad (3.63)$$

which depends only on m, the rank of the p-form field strength, F, and two constants $b_{1,2}$. (One example of this can be seen in eq. (4.8), in the context of the five-dimensional supergravity that is the starting point of the dimensional reduction in chapter 4.)

I will present here only a special example – the case A-split that has been the focus of attention in the previous section, with a Maxwell two-form, $m=2$. The contribution to $\Delta_S\Psi_M$ by an expression of the form (3.63) is, all things considered,

$$\begin{aligned}\delta_S\Psi_\mu \;=\; & \ldots + \Delta^r B_\mu{}^m\Xi_m\epsilon' + b_1\Delta^{r+s}e'_\mu{}^\delta\Big[\gamma^{\alpha\beta}{}_\delta\hat{\Gamma}^v F_{\alpha\beta} - 2\hat{\Gamma}^b\gamma^\alpha{}_\delta F_{\alpha b} + \epsilon_v\gamma_\delta\hat{\Gamma}^v\hat{\Gamma}^{ab}F_{ab} \\ & - b_2\gamma^\beta\hat{\Gamma}^v F_{\delta\beta} - b_2\hat{\Gamma}^b F_{\delta b}\Big] \end{aligned} \quad (3.64)$$

and

$$\delta_S\Psi_m = \ldots + \Delta^r\Xi_m\epsilon', \quad (3.65)$$

where

$$\begin{aligned}\Xi_m \;=\; & b_1\Big[\epsilon_v\gamma^{\alpha\beta}\hat{\Gamma}_d F_{\alpha\beta} + 2\gamma^\alpha\hat{\Gamma}^v\hat{\Gamma}^b{}_d F_{\alpha b} + \hat{\Gamma}^{ab}{}_d F_{ab} \\ & - b_2\gamma^\beta\hat{\Gamma}^v F_{d\beta} - b_2\hat{\Gamma}^b F_{db}\Big]e_m{}^d \end{aligned} \quad (3.66)$$

and where the dots indicate the contribution from the space-time covariant derivative, already taken care of in the previous section. Using eq. (3.59), one can find the corresponding contribution to $\delta_S \Psi'_\mu$, to wit:

$$\begin{aligned}\delta\Psi'_\mu \;=\; \ldots + \Delta^s \epsilon_v b_1 e'_\mu{}^\delta \Big\{ & F_{\alpha\beta}\hat{\Gamma}^v \gamma^{\alpha\beta}{}_\delta (1 - sd_2) - F_{\delta\beta}\gamma^\beta \hat{\Gamma}^v (b_2 + 2sd_2) \\ & + F_{\alpha b}\hat{\Gamma}^b \gamma^\alpha{}_\delta [s(2d_2 - b_2) - 2(1+s)] \\ & + F_{\delta b}\hat{\Gamma}^b [(2 + b_2)s - (b_2 + 2sd_2)] \\ & + \epsilon_v F_{ab}\hat{\Gamma}^v \hat{\Gamma}^{ab}\gamma_\delta [s(2 - d_2) + 1 + sb_2] \Big\}. \end{aligned} \tag{3.67}$$

3.8 Supersymmetry transformations of the new spin-1/2 field

To close, for the supersymmetry transformations of the new spin-1/2 field. The basic plan of action is the same as in the preceding section, and the starting point will once more be the rewriting of the variation of $\tilde{\chi}_a$ in terms of that of Ψ_m,

$$\delta_{S'}\tilde{\chi}_a = \Delta^{s/2} e_a{}^m \, (\delta_{S'}\Psi_m) + \lambda \frac{s}{2} (\overline{\epsilon}' \hat{\Gamma}^v \hat{\Gamma}^c \tilde{\chi}_c)\tilde{\chi}_a - \lambda (\overline{\epsilon}' \hat{\Gamma}^v \hat{\Gamma}^c \tilde{\chi}_a)\tilde{\chi}_c. \tag{3.68}$$

Again, there is a variety of contributions to the full variation to consider; let us, again, focus only on the omnipresent contribution $\Delta^{s/2} e_a{}^m \, (D_m \epsilon)$ and one example for a p-form contribution.

Contribution of $D_m\epsilon$

It is straightforward to find the contribution of $D_m\epsilon$. Ignoring, as ever, the contribution of contorsion terms, one has but to use the redefinition $D_m(\Delta^r \epsilon')$ and insert the split version of $\bar{D}_m$ given in eq. (3.50) as, with no Lorentz vector indices involved, $\bar{D}_m$ is indistinguishable from D_m. As a result,

$$\begin{aligned}\Delta^r e_a{}^m (D_m\epsilon) \;=\; \Big\{ & \Delta^s e_a{}^n \Big[\partial_n + \frac{s}{2}(\partial_n \ln \Delta)\Big] + \frac{1}{8}(\epsilon_v \Delta^{-s}\Omega'_{\delta\varepsilon a} - 2\Delta^s \Omega'_{a[\delta\varepsilon]})\gamma^{\delta\varepsilon} \\ & + \frac{1}{2}\hat{\Gamma}^v \hat{\Gamma}^b \gamma^\alpha \Omega'_{\alpha(ab)} + \frac{1}{8}\Delta^s(\Omega'_{dea} - 2\Omega'_{a[de]})\hat{\Gamma}^{de}\Big\}\epsilon'. \end{aligned} \tag{3.69}$$

Once more, the case $d_1 = 3$ has an especially simple structure – at least in the case of dimensional reduction, in which terms proportional to some derivative ∂_n (such as $\Omega'_{a\delta\varepsilon}$ or Ω'_{dea}) do not contribute. In that case, one can again dualize as prescribed in eq. (3.53), namely $\Omega'_{\delta\varepsilon c}\gamma^{\delta\varepsilon} = -2(\epsilon_3)^{-1}\gamma^\alpha \Omega'_{\alpha c}$. After dualization, all remaining terms have the same space-time structure, namely $\gamma^\alpha \epsilon' V_\alpha$ with the V_α some flat space-time vector coefficients.

Contribution of a 2-form field-strength *F*

With the generic form for the contribution of a two-form field F to the supersymmetry variation of a gravitino given in (3.63), there is a contribution $\Delta^s {e_a}^m \Xi_m \epsilon'$ to the matter fermion's supersymmetry variation, with Ξ_m the expression defined in (3.66), explicitly:

$$b_1 \Delta^s \Big\{ \epsilon_v \gamma^{\alpha\beta} \hat{\Gamma}_a F_{\alpha\beta} + \hat{\Gamma}^v (2{\hat{\Gamma}^c}_a + b_2 \delta^c_a) \gamma^\alpha F_{\alpha c} + ({\hat{\Gamma}^{cd}}_a + b_2 \hat{\Gamma}^c \delta^d_a) F_{cd} \Big\} \epsilon'. \qquad (3.70)$$

In the case of dimensional reduction, the terms F_{cd} do not contribute. For $d_1 = 3$, the remaining terms again can be given the same gamma matrix structure – that of containing a single γ^α. All that is needed is the dualization of $F_{\alpha\beta}$ to

$$F_\delta = \frac{1}{2} {\varepsilon_\delta}^{\alpha\beta} F_{\alpha\beta} \quad \Leftrightarrow \quad F_{\alpha\beta} = (-)^q {\varepsilon_{\alpha\beta}}^\delta F_\delta, \qquad (3.71)$$

which entails $\gamma^{\alpha\beta} F_{\alpha\beta} = -2(\epsilon_3)^{-1} \gamma^\delta F_\delta$.

Chapter 4

The dimensional reduction from five to three dimensions

Now, with the general expressions for the dimensional split collected in chapter 3, and with a suitable point of reference in the form of the three-dimensional model with $G_{2(+2)}/SO(4)$-structure, constructed in chapter 2, the preparations are complete. Putting those results to work, this chapter deals with the dimensional reduction of $d = 5, \mathcal{N} = 2$ supergravity, and with its proper reformulation in order to expose the hidden $G_{2(+2)}/SO(4)$ symmetry structure.

The chapter is structured as follows: The first section, 4.1, introduces the starting point of our quest, the $\mathcal{N} = 2$ supergravity in five dimensions. Section 4.2 deals with the crucial matters of spinorial splits and the implementation of the enhanced local symmetry. Then, it is time to match the reduced Lagrangian and the supersymmetry variations with the three-dimensional model, in order to recover objects such as P_μ and the Q_μ. These entities are derived in section 4.3, while additional cross-checks arising from the comparison of the two models form the content of section 4.4.

4.1 The N=2 supergravity in five dimensions

The $\mathcal{N} = 2$ supersymmetry in five dimensions, first constructed in 1980 [28, 18][1], contains as its basic ingredients the gravity multiplet made up of the vielbein (in this case: fünfbein) $E_M{}^A$, a Maxwell vector field A_M, and the gravitino Ψ_M. There are no ordinary Majorana spinors in this space-time and indeed there is no $\mathcal{N} = 1$ supergravity – the minimal construction is that with a symplectic (or pseudo-) Majorana condition, giving the gravitino an extra index $i = 1, 2$. This leads to the $\mathcal{N} = 2$ model reviewed here: The five-dimensional vielbein has $5(5-3)/2 = 5$ physical degrees of freedom. A symplectic Majorana spinor, its components doubled by the extra index yet simulta-

[1] While these are the original works on the subject, I have not taken my data about the supergravity from there, but started from the generic first order supergravity Lagrangian in [127], adapting it to the case in hand and explicitly checking its supersymmetry invariance in the process.

neously halved by the reality condition, has $2 \cdot 2^{[5/2]}/2 = 4$ independent components, giving the gravitino $4 \cdot (5-3) = 8$ physical degrees of freedom. The difference between bosonic and fermionic degrees of freedom, 3 in favour of the fermionic side, must be made up; a photon with $5-2=3$ physical degrees of freedom fills the gap.

Spinorial product and Fierz identities

First for the spinors and gamma matrices. Let our starting point be the complexified Clifford algebra $\mathcal{C}_c(1,4)$, in other words, the gamma matrices with signature $(+,-,-,-,-)$. Let Γ^A be the gamma matrices forming an irreducible representation with $\Gamma^v = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4 = \epsilon_3\mathbb{1}$. With these preparations, the natural choice is a symmetric scalar product for the complex spinors, which is conjugate-linear and whose adjoint is Clifford reversion,

$$(\overline{\psi}\Gamma^{A_1}\cdots\Gamma^{A_m}\phi)^* = (\overline{\phi}\Gamma^{A_m}\cdots\Gamma^{A_1}\psi). \tag{4.1}$$

From these algebraic requirements, all other important properties follow; however, for the concreteness physicists are most comfortable with, we can also choose an explicit basis $\Gamma^0 = -\epsilon_3\sigma_2\otimes\sigma_2, \Gamma^1 = i\sigma_1\otimes\sigma_2, \Gamma^2 = i\sigma_3\otimes\sigma_2, \Gamma^3 = i\mathbb{1}\otimes\sigma_1, \Gamma^4 = i\mathbb{1}\otimes\sigma_3$, where the σ_i are the customary Pauli matrices.

Let C_+ be the complex conjugation matrix[2], satisfying $(\Gamma^A)^* = C_+\Gamma^A C_+{}^{-1}$ (in the explicit basis given above, let us make the choice $C_+ = \Gamma^3\Gamma^4 = \mathbb{1}\otimes i\sigma_2$). It has the properties $(C_+)^*C_+ = -\mathbb{1}$ and $(\Gamma^0)^{-1}(C_+)^\dagger(\Gamma^0)^*C_+ = +\mathbb{1}$. Evidently, this makes it impossible to define real Majorana spinors (with a real Majorana condition formulated with such a C_+, complex conjugation would not be an involution); however, it is possible to define symplectic ones. Let Ω_{ij} be the two-dimensional symplectic metric and Ω^{ij} minus its inverse (both are antisymmetric and, for definiteness, let us choose the standard form equal to the two-dimensional epsilon symbol, $\Omega_{12} = \Omega^{12} = +1$). Each spinor shall carry a symplectic index and fulfil a symplectic Majorana condition

$$(\psi^*)_i = -\Omega_{ij}C_+\psi^j \tag{4.2}$$

where a convention has been chosen whereby complex conjugation automatically moves lower symplectic indices up, and upper indices down[3]. Combining this Majorana condition with the symmetry of the spinor product, we see that

$$\begin{aligned}(\overline{\psi}_i\Gamma^{A_1}\cdots\Gamma^{A_m}\phi^j) &= -\Omega_{ik}\Omega^{jl}(\overline{\phi}_l\Gamma^{A_m}\cdots\Gamma^{A_1}\psi^k)\\ &= (\overline{\phi}_i\Gamma^{A_m}\cdots\Gamma^{A_1}\psi^j) - \delta_i^j(\overline{\phi}_k\Gamma^{A_m}\cdots\Gamma^{A_1}\psi^k)\end{aligned} \tag{4.3}$$

[2]Related to the more common charge conjugation matrix C by $C_+ = (\Gamma^{0T})^{-1}C$.

[3]Some papers, among them the original publication by Cremmer, choose a different convention where the symplectic metric is used to raise and lower indices, $\psi_i := \Omega_{ij}\psi^j$, and spinor conjugation is taken to include an application of the symplectic metric, as well, namely $\overline{\psi}^i := \Omega_{ij}\,(\psi^j)^\dagger\Gamma^0$. Using such conventions, a product $(\overline{\psi}^i\chi_i)$ would correspond to direct contraction $(\overline{\psi}_i\chi^i)$ in the convention used here.

and in particular, for a direct contraction of symplectic indices,

$$(\overline{\psi}_i\Gamma^{A_1}\cdots\Gamma^{A_m}\phi^i) = -(\overline{\phi}_i\Gamma^{A_m}\cdots\Gamma^{A_1}\psi^i). \tag{4.4}$$

Also, it follows that such directly contracted spinor products are purely imaginary. Associated with the symplectic index is an invariance under a global $\mathfrak{sp}(1) \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3)$ symmetry. Taking into account the duality relations between the gamma matrices, the basic Fierz identity in five dimensions is

$$\Psi\overline{\Phi} \;\; = \;\; -\frac{1}{4}\left[(\overline{\Phi}\Psi) + (\overline{\Phi}\Gamma_A\Psi)\Gamma^A - \frac{1}{2}(\overline{\Phi}\Gamma_{AB}\Psi)\Gamma^{AB}\right].$$

This identity is of use in calculating the commutators of supersymmetry variations acting on fermions. In order to show the supersymmetry invariance of the Lagrangian, yet another Fierz identity is needed, namely

$$\Gamma_{A[M}\Psi^i_N(\overline{\Psi}_{|j|P}\Gamma^A\Psi^j_{Q]}) + \Gamma_{[M}\Psi^i_N(\overline{\Psi}_{|j|P}\Psi^j_{Q]}) - \Psi^i_{[M}(\overline{\Psi}_{|j|N}\Gamma_P\Psi^j_{Q]}) = 0, \tag{4.5}$$

which can be derived by successive application of (4.5)[4].

Lagrangian and supersymmetry variations

The theory has a Lagrangian density consisting of the usual Einstein-Hilbert, Rarita-Schwinger and Maxwell kinetic terms, plus a Chern-Simons term, additional quadratic interaction terms between the gravitino and the Maxwell field, and higher fermionic terms, concretely[5]

$$\begin{aligned}\mathcal{L}_{5|2} \;\; = \;\; & -\frac{1}{4\kappa^2}E\mathcal{R} - \frac{1}{4}EF_{MN}F^{MN} - \epsilon_3\frac{1}{6\sqrt{3}}(E\varepsilon^{MNPQR})\kappa A_M F_{NP}F_{QR} \\ & +\frac{i}{2}E(\overline{\Psi}_{Mi}\Gamma^{MNP}D_N(\omega)\Psi^i_P) + \frac{\sqrt{3}}{8}i\kappa EF_{MN}[2(\overline{\Psi}^M_i\Psi^{Ni}) + (\overline{\Psi}_{Pi}\Gamma^{MNPQ}\Psi^i_Q)] \\ & +\frac{\kappa^2}{16}E(\overline{\Psi}_{Mi}\Psi^i_N)[3(\overline{\Psi}^M_j\Psi^{Nj}) + 2(\overline{\Psi}_{Pj}\Gamma^{MNPQ}\Psi^j_Q)],\end{aligned} \tag{4.6}$$

where F_{MN} is the usual field strength $F_{MN} := 2\partial_{[M}A_{N]}$ of the Maxwell field A_M. Total derivatives aside, this Lagrangian is invariant under the supersymmetry variations

$$\begin{aligned}\delta E_M{}^A \;\; &= \;\; i\kappa^2(\overline{\epsilon}_i\Gamma^A\Psi^i_M), \\ \delta\Psi^i \;\; &= \;\; \hat{D}(\omega, F)\epsilon^i, \\ \delta A \;\; &= \;\; \frac{\sqrt{3}}{2}\,i\kappa\,(\overline{\epsilon}_i\Psi^i),\end{aligned} \tag{4.7}$$

[4]Details of this derivation have been deposited in the appendix in section A.3

[5]Again, in the conventions used, the curved-index epsilon symbol ε^{MNPQR} is defined by applying vielbeins and metric coefficients to the all-lower flat-index epsilon symbol ε_{ABCDE} which, in its turn, is defined by $\varepsilon_{12345} = +1$.

where the supercovariant derivative $\hat{D}$ is defined by

$$\hat{D}_M(\omega, F)\epsilon := D_M(\omega)\epsilon + \frac{1}{4\sqrt{3}}\kappa(\Gamma_{ABC} + 4\eta_{BC}\Gamma_A)F^{AB}E_M{}^C\epsilon. \tag{4.8}$$

4.2 Reduction of spinorial quantities and the enhanced local symmetry

In identifying the hidden local symmetry, much depends on the reduction of gamma matrices and spinors. In the split $5 \to 3+2$, the space-time is odd- and the internal space even-dimensional, leading to what was called "case A" in section 3.3,

$$\Gamma^A = \begin{cases} \gamma^\alpha \otimes \hat{\Gamma}^v \\ \mathbb{1} \otimes \hat{\Gamma}^a. \end{cases} \tag{4.9}$$

The Γ^A form the "mostly minus" Clifford algebra $\mathcal{C}(1,4)$, and for the three-dimensional space to have the signature of a space-time, the only possibility is the tensor decomposition $\mathcal{C}(1,4) = \mathcal{C}(2,1) \otimes \mathcal{C}(0,2)$, in other words: space-time has a "mostly plus" metric with signature $(-,+,+)$, and internal space-time has a Euclidean signature $(-,-)$. For concreteness, let us choose

$$\gamma^0 = i\epsilon_3\sigma_2, \quad \gamma^1 = \sigma_1, \quad \gamma^2 = \sigma_3$$

as space-time gamma matrices and

$$\hat{\Gamma}^1 = i\sigma_1, \quad \hat{\Gamma}^2 = i\sigma_3 \quad \Rightarrow \quad \hat{\Gamma}^v = i\sigma_2$$

for internal space. (With these choices, the product Clifford algebra in five dimensions is the concrete representation chosen before, in section 4.1.)

Next, for the reality properties inherited from the product algebra. In section 4.1, the complex conjugation matrix chosen was $C_+ = \Gamma^3\Gamma^4 = \mathbb{1} \otimes i\sigma_2$, from which can be read off the reality conditions: The matrices γ^α can be chosen real, just as they were chosen above; this corresponds to the fact that there is a real Clifford algebra $\mathcal{C}(2,1)$ isomorphic to a matrix algebra over the reals. For the matrices $\hat{\Gamma}^a$, the complex conjugation matrix is σ_2, corresponding to the possibility of a symplectic, but not an ordinary Majorana reality condition and to the fact that $\mathcal{C}(0,2)$ is isomorphic to the quaternions (more will be said about this in chapter 5).

Properties of the internal matrices $\hat{\Gamma}$

First, for a look at a few properties of the $\hat{\Gamma}$, noting right away that one property important for the reduction formulae from chapter 3 is $(\hat{\Gamma}^v)^2 = \epsilon_v\mathbb{1} = -\mathbb{1}$. The $\hat{\Gamma}$ are $SO(2)$-gamma matrices, yet it is straightforward to make the transition to a $SO(3)$:

Including the Clifford volume element $\hat{\Gamma}^v$, it is possible to form a larger algebra made up of $\hat{\Gamma}^r = (\hat{\Gamma}^a, \hat{\Gamma}^v)$ with $r = 1, \ldots, 3$, consisting of the Pauli matrices with an additional imaginary factor and thus satisfying

$$\hat{\Gamma}^r\hat{\Gamma}^s = \varepsilon^{rst}\hat{\Gamma}_t + \eta^{rs}\mathbb{1}_2,$$

with η^{rs}, the all-negative Euclidean metric, the natural extension of the (negative) $SO(2)$ metric η^{ab}, and with $\varepsilon^{123} = -1$. This is just the algebra relation for $\mathfrak{su}(2)$. From the complex conjugation matrix given above, it follows that the matrices $\hat{\Gamma}^r$ satisfy the reality condition

$$((\hat{\Gamma}^r)^{\bar{a}}{}_{\bar{b}})^* = -\varepsilon_{\bar{a}\bar{c}}\,(\hat{\Gamma}^r)^{\bar{c}}{}_{\bar{d}}\,\varepsilon^{\bar{d}\bar{b}}, \tag{4.10}$$

which, in the conventions used here, is the reality condition the $\mathfrak{so}(3)$-generators in the decomposition of the $G_{2(+2)}$ must fulfil (compare B).

For the Clifford algebra of the $\hat{\Gamma}^r$, one can derive the basic Fierz identity

$$\psi\overline{\phi} = -\frac{1}{2}\left[(\overline{\phi}\psi) + (\overline{\phi}\hat{\Gamma}^t\psi)\hat{\Gamma}_t\right], \tag{4.11}$$

(where duality relations for the gamma matrices $\hat{\Gamma}^r$ have been used, namely $\hat{\Gamma}^{rst} = \varepsilon^{rst}$ and $\varepsilon_{rst}\hat{\Gamma}^{st} = -2\hat{\Gamma}_r$). This, in turn, can be used to derive another Fierz identity,

$$\hat{\Gamma}^r\psi\overline{\phi}\hat{\Gamma}^s = -\frac{1}{2}\eta^{rs}\left((\overline{\phi}\psi) - (\overline{\phi}\hat{\Gamma}_t\psi)\hat{\Gamma}^t\right) - (\overline{\phi}\hat{\Gamma}^{(r}\psi)\hat{\Gamma}^{s)} + \frac{1}{2}\left((\overline{\phi}\hat{\Gamma}^{rs}\psi) - (\overline{\phi}\psi)\hat{\Gamma}^{rs}\right)$$

(where, once more, duality relations for the $\hat{\Gamma}$ were taken into account). From this identity, we can read off at least two more: Contracting r, s with the metric, and introducing $SO(2)$ spinor indices $\bar{a}, \bar{b}, \ldots$, there is

$$(\hat{\Gamma}^r)^{\bar{a}}{}_{\bar{b}}(\hat{\Gamma}_r)^{\bar{c}}{}_{\bar{d}} = 2\delta^{\bar{a}}_{\bar{d}}\delta^{\bar{c}}_{\bar{b}} - \delta^{\bar{a}}_{\bar{b}}\delta^{\bar{c}}_{\bar{d}}, \tag{4.12}$$

while contraction with the epsilon tensor yields the identity

$$\varepsilon_{rst}(\hat{\Gamma}^s)^{\bar{a}}{}_{\bar{b}}(\hat{\Gamma}^t)^{\bar{c}}{}_{\bar{d}} = \delta^{\bar{a}}_{\bar{d}}(\hat{\Gamma}_r)^{\bar{c}}{}_{\bar{b}} - \delta^{\bar{c}}_{\bar{b}}(\hat{\Gamma}_r)^{\bar{a}}{}_{\bar{d}} \tag{4.13}$$

(both of these can easily be rewritten as identities between the $SO(2)$-Clifford matrices, inserting $\varepsilon^{vab} = -\varepsilon^{ab}$). One consequence of this can be seen by introducing the all-positive flat metric $\eta_{\bar{a}\bar{b}}$, which is invariant under the group Spin(2) (generated by $\hat{\Gamma}^v$). Employing it to lower indices, and using the fact that $(\hat{\Gamma}^v)_{\bar{a}\bar{b}} = \varepsilon_{\bar{a}\bar{b}}$ for the totally antisymmetric epsilon symbol with $\varepsilon_{12} = 1$, it can be seen that the $\hat{\Gamma}^a$ satisfy not only their usual Clifford relation

$$(\hat{\Gamma}^a)^{\bar{a}}{}_{\bar{b}}(\hat{\Gamma}^b)^{\bar{b}}{}_{\bar{c}} + (\hat{\Gamma}^b)^{\bar{a}}{}_{\bar{b}}(\hat{\Gamma}^a)^{\bar{b}}{}_{\bar{c}} = 2\eta^{ab}\delta^{\bar{a}}_{\bar{c}}, \tag{4.14}$$

but also a Clifford relation in which the role of two-dimensional vector and spinor indices is partially reversed, namely

$$(\hat{\Gamma}^a)_{\bar{a}\bar{b}}(\hat{\Gamma}_a)_{\bar{c}\bar{d}} + (\hat{\Gamma}^a)_{\bar{d}\bar{b}}(\hat{\Gamma}_a)_{\bar{c}\bar{a}} = 2\eta_{\bar{a}\bar{d}}\eta_{\bar{b}\bar{c}}. \tag{4.15}$$

This is reminiscent of similar relations for the $SO(8)$, which are associated with the triality property of $SO(8)$ (used in the construction of the $E_{8(+8)}/SO(16)$ model). Let us derive one other identity to be used later on: Applying (4.12) on both sides of

$$(\hat\Gamma^a)^{\bar e}{}_{\bar h}(\hat\Gamma^v)^{\bar a}{}_{\bar b}(\hat\Gamma^d)^{\bar h}{}_{\bar a}(\hat\Gamma_d)^{\bar d}{}_{\bar g} = (\hat\Gamma^v\hat\Gamma^a)^{\bar e}{}_{\bar h}(\hat\Gamma^d)^{\bar h}{}_{\bar b}(\hat\Gamma_d)^{\bar d}{}_{\bar g},$$

shows that

$$(\hat\Gamma^a)^{\bar e}{}_{[\bar b}(\hat\Gamma^v)^{\bar d}{}_{\bar g]} = (\hat\Gamma^v\hat\Gamma^a)^{\bar e}{}_{[\bar b}\delta^{\bar d}_{\bar g]}. \tag{4.16}$$

From the old to the new gravitino

Schematically, the dimensional reduction of the gravitino field Ψ_M of the five-dimensional theory proceeds as

$$\Psi^i_A \quad \begin{array}{l} \nearrow\ \Psi^{i\bar a}_\alpha \\ \searrow\ \Psi^{i\bar a}_a \end{array}$$

as the Lorentz vector index A gets split into a three-dimensional space-time index α and an internal index a, while the five-dimensional spinor index is split into a combination of three-dimensional spinor index and internal spinor index $\bar a$ (in order not to increase the number of different indices, five-dimensional space-time spinor indices on the left hand and three-dimensional space-time spinor indices on the right hand side have been suppressed).

We have seen in the preceding chapter that the split will also involve a redefinition (3.37) in order to arrive at the new gravitino ${\Psi'}_\mu^{i\bar a}$. The internal index structure, however, remains the same, and this is the focus of the present section: The only internal index of ${\Psi'}_\mu^{i\bar a}$ is $\bar a$. So far, this is an index of the Spin$(2) \simeq U(1)$ whose generator is $\hat\Gamma^1\hat\Gamma^2 = \hat\Gamma^v$. But it is readily seen that the symmetry group associated with the spinor product $\phi^\dagger\chi$ of two SO(2)-spinors ϕ and χ is larger than that – in fact, by definition the maximal symmetry of $\phi^\dagger\chi = (\phi^*)_{\bar a}\chi^{\bar a}$ is $U(2) \simeq U(1)\times SU(2)$, with an infinitesimal transformation given by

$$(\delta\phi)^{\bar a} = (\epsilon_r\hat\Gamma^r + \epsilon_0(i\mathbb{1}))^{\bar a}{}_{\bar b}\,\phi^{\bar b}, \tag{4.17}$$

where the real parameters ϵ_r, $r = 1,2,3$ parametrize an $SU(2)$ transformation using the enlarged set of gamma matrices $\hat\Gamma^r = (\hat\Gamma^a, \hat\Gamma^v)$, defined in the previous section, and the real parameter ϵ_0 parametrizes the $U(1)$ generated by $i\mathbb{1}$.

However, there is a further matter to consider. For fully complex entities $\phi^{\bar a}$, the above analysis would have been sufficient, but the new gravitino has inherited a reality condition from its five-dimensional counterpart: the symplectic Majorana condition (4.2). The only symmetry transformations allowed are those that respect this condition – transformed spinors must still be symplectic Majorana spinors. Dropping the Lorentz index α, the reality condition (4.2) can be rewritten for the new gravitino as

$$({\Psi'}_\mu^{\,*})_{i\bar a} = -\varepsilon_{ij}\varepsilon_{\bar a\bar b}{\Psi'}_\mu^{j\bar b}, \tag{4.18}$$

where the fact has been used that the complex conjugation matrix $i\sigma_2$ can be rewritten as $(i\sigma_2)_{\bar{a}\bar{b}} = \varepsilon_{\bar{a}\bar{b}}$; also, let the convention for the indices $\bar{a}, \bar{b}, \ldots$ be like the one already in force for the indices i, j; namely, that complex conjugation lowers upper indices, and raises lower ones. The $U(1)$ transformations, however, do not (as, obviously, $i\mathbb{1}\cdot\sigma_2 \neq -i\sigma_2\cdot\mathbb{1}$), and will have to be dropped. The only enhancement of the symmetry allowed by the spinor product is thus the enlargement of spin(2) to $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$. With this enhancement in place, the transformation and reality properties of the new gravitino Ψ'_μ are exactly those of the gravitino of the three-dimensional model.

From the old to the new spin 1/2-fermions

Having taken care of the new gravitino, let us turn to the spin-1/2 fermions that arise from $\Psi^i_A \rightarrow \Psi^{i\bar{a}}_a$. Again, the split involves some redefinition – as we have seen in section 3.5, it is advisable to use instead of $\Psi^{i\bar{a}}_a$ redefined fields

$$\chi^{i\bar{a}}_a := \Delta^{-1/2}\left(\Psi^{i\bar{a}}_a + \overline{c}(\hat{\Gamma}_a\hat{\Gamma}^c)^{\bar{a}}{}_{\bar{b}}\Psi^{i\bar{b}}_c\right), \tag{4.19}$$

(cf. eq. (3.44)) with some as yet unfixed constant $\overline{c}$. But even with the redefinition, the index structure $\chi^{i\bar{a}}_a$ remains the same, and that alone gives pause, thinking back to the basic group theoretical considerations for the three-dimensional model that were laid down in section 2.2: There, the gravitino's index structure was $\Psi^{i\bar{a}}$, with the i-index of the $\mathfrak{so}(3)_F$ and the $\bar{a}$-index of the $\mathfrak{so}(3)_B$. But the index structure of the spin-1/2 fermion was $\chi^{i\dot{a}\dot{b}\dot{c}}$, with indices $\dot{a}, \dot{b}, \dot{c}$ not of the $\mathfrak{so}(3)_B$, but of the additional group $\mathfrak{so}(3)_2$. Thus, we are dealing with an even further enlargement of the local symmetries: To reach the local symmetries of the full three-dimensional $G_{2(+2)}/SO(4)$ model, not only is it necessary to pass from Spin(2) to $SO(3)$, as was done in the preceding section; it is also necessary to assume that this $SO(3)$, as it originally occurs in the dimensionally reduced theory, is only the diagonal subgroup of a larger $SO(3)_B \times SO(3)_2$. The practical implementation of this is as follows: Start by treating the enhanced symmetry of the matter fermions, to be developed below, as belonging to a different $SO(3)_2$, whose indices are denoted by $\dot{a}, \dot{b}, \dot{c}$. This is possible because no direct contraction of these indices with the $\bar{a}, \bar{b}, \ldots$ arises from the dimensional reduction – if there were hypothetical terms like $\overline{\Psi}_{\alpha i\bar{a}}\Psi^{j\bar{a}}_a$, one would be hard-pressed to disentangle this direct index construction, assigning only indices $\dot{a}, \dot{b}, \dot{c}$ to the spin 1/2-fermion and yet an index $\bar{a}$ to the gravitino. Wherever a combination of Ψ_a and Ψ_α arises, it will always feature compound objects such as $P^{\bar{a}\dot{a}\dot{b}\dot{c}}$ providing the proper index structure. There will still remain the fact that, inside these compound objects, there will be contained manifestly non-invariant constructs such as $\delta^{\bar{a}}_{\dot{a}}$; however, that is no different from, for instance, the $E_{8(+8)}/SO(16)$ construction: Here as there, the entities arising from the dimensional reduction are gauge-fixed with respect to the local symmetry group. In our case, dimensional reduction leads to a “diagonal gauge” for the local symmetry $SO(3)_B \times SO(3)_2$.

Even with these preliminaries out of the way, there still remains the question of what representation to assign to the spin 1/2-fermions. The answer is shown by reference to the three-dimensional model: The four components associated with the $SO(2)$-indices a and $\dot{a}$ in the expression $\chi_a^{i\dot{a}}$ should fall into the **4**-representation of the enlarged $SO(2) \to SO(3)$, corresponding to an object $\chi^{i\dot{a}\dot{b}\dot{c}}$ totally symmetric in $\dot{a}, \dot{b}, \dot{c}$. This can be constructed with the help of the matrices $(\hat{\Gamma}^a)^{\dot{a}}{}_{\dot{b}}$ defined before (albeit with the indices named differently), and with the totally antisymmetric $\varepsilon^{\dot{a}\dot{b}}$ with $\varepsilon^{12} = 1$, defining

$$\chi^{i\dot{a}\dot{b}\dot{c}} = \chi_a^{i(\dot{a}}(\hat{\Gamma}^a)^{\dot{b}}{}_{\dot{d}}\varepsilon^{\dot{c})\dot{d}}. \tag{4.20}$$

The reverse map for use when substituting the new $\chi^{i\dot{a}\dot{b}\dot{c}}$ for the old $\chi_a^{i\dot{a}}$ is given by

$$\chi_a^{i\dot{c}} = \frac{1}{2}(\delta_a^c \delta_{\dot{e}}^{\dot{c}} + (\hat{\Gamma}_a \hat{\Gamma}^c)^{\dot{c}}{}_{\dot{e}})(\hat{\Gamma}_c)^{\dot{d}}{}_{\dot{a}}\chi^{i\dot{a}\dot{b}\dot{e}}\varepsilon_{\dot{b}\dot{d}}. \tag{4.21}$$

This can be derived using (4.12). So far, the derivation hasn't bothered with reality conditions, and we shall need to make up for that omission now – starting as far back as the redefinition (4.19), where use of the reality condition for the $\hat{\Gamma}^a$ and of the fact that $(i\sigma_2)_{\dot{a}\dot{b}} = \varepsilon_{\dot{a}\dot{b}}$ shows that, yes, even the redefined $\chi_a^{i\dot{a}}$ inherit the symplectic Majorana condition $(\chi_a^*)_{i\dot{a}} = -\varepsilon_{ij}\varepsilon_{\dot{a}\dot{b}}\chi_a^{j\dot{b}}$. Direct calculation shows that in the same way, $\chi^{i\dot{a}\dot{b}\dot{c}}$ given by (4.20) inherits a suitable symplectic Majorana condition

$$(\chi^*)_{i\dot{a}\dot{b}\dot{c}} = -\varepsilon_{ij}\varepsilon_{\dot{a}\dot{d}}\varepsilon_{\dot{b}\dot{e}}\varepsilon_{\dot{c}\dot{f}}\chi^{j\dot{d}\dot{e}\dot{f}}. \tag{4.22}$$

Next, let us recall the reason that a redefinition such as (4.19) was considered in section 3.5 in the first place. In that section it was shown how the five-dimensional Rarita-Schwinger term gives rise to a kinetic term for the spin-1/2 fermions: a term quadratic in Ψ_a, with a derivative $\gamma^\mu\partial_\mu$ acting on one of the factors. However, as far as the internal spinor structure was concerned, the term resulting from the dimensional reduction did not have the simple diagonal form desirable for a kinetic term; instead, adapting (3.44) to the case at hand, it had the structure

$$(\overline{\Psi}_a^{i\dot{a}}\left[\eta^{ac}\delta_{\dot{b}}^{\dot{a}} + (\hat{\Gamma}^a\hat{\Gamma}^c)^{\dot{a}}{}_{\dot{b}}\right]\gamma^\nu\partial_\nu\Psi_c^{i\dot{b}}).$$

With this in mind, let us now look at sesquilinear contractions between objects defined as in (4.20). One basic identity is

$$(\chi^*)_{i\dot{a}\dot{b}\dot{c}}\phi^{i\dot{a}\dot{b}\dot{c}} = \frac{2}{3}(\chi_a^*)_{i\dot{a}}[-2\overline{\eta}^{ab}\delta_{\dot{b}}^{\dot{a}} + (\hat{\Gamma}^v)^{\dot{a}}{}_{\dot{b}}\varepsilon^{ab}]\phi_b^{i\dot{b}}. \tag{4.23}$$

Using this, and choosing either $\overline{c} = -2$ or $\overline{c} = +1$, the kinetic term indeed takes on the simple diagonal form

$$(\overline{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\mu\partial_\mu\chi^{i\dot{a}\dot{b}\dot{c}}). \tag{4.24}$$

With this manifestly $SO(3)$-invariant expression, and with the Majorana condition (4.22), the symmetry enhancement on the spin 1/2-fermions is complete, and the derivation has made contact with the spin 1/2-fermions of the three-dimensional model. The generators of the $SO(3)$ are, again, given by the gamma matrices $\hat{\Gamma}^a, \hat{\Gamma}^v$.

4.3 Matching the models I: Establishing the correspondence

Now that the dimensionally reduced gravitino and spin 1/2-fermion have been matched up with their three-dimensional counterparts from the $G_{2(+2)}/SO(4)$ coset model constructed in chapter 2 (hereinafter referred to in an abbreviated fashion as the "three-dimensional model"), it is time to match Lagrangian and supersymmetry variations, as well. This section is devoted to the determination of three-dimensional quantities – the goal is to fix the remaining free parameters (such as ϵ_1 and the ambiguity in the choice of $\overline{c}$), and also to identify the compound entities: the scalar term $(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}$ and the connection coefficients $(Q_\mu)^{\bar{a}}{}_{\bar{b}}$ and $(Q_\mu)_{\dot{a}}{}^{\dot{b}}$ for the $SO(3)_B$ and the $SO(3)_2$, respectively. With all these quantities fixed and no further freedom of choice, the remaining matches, which play the role of consistency checks, will be scrutinized in section 4.4.

Matching the pure supergravity

To get our bearings, let us start with the pure supergravity part of the reduced theory. The dreibein $e'_\mu{}^\alpha$ is mapped to what, in the three-dimensional model, we have written without the prime, $e_\mu{}^\alpha$; the determinant e' to e, the metric $g^{\mu\nu}$ to $g'^{\mu\nu}$, and so on. By (3.18), the Einstein-Hilbert-term $E\mathcal{R}$ reduces to $\epsilon_v e'\mathcal{R} = -e'\mathcal{R}$. In order to match this and the associated coefficient of the Lagrangian (4.6) with its three-dimensional counterpart in (2.38), it is necessary to give the whole Lagrangian of the dimensionally reduced theory an additional minus sign.

Next, we can fix the parameter ϵ_1 by looking at the $D_\mu\epsilon$-part of the new gravitino's supersymmetry variation. By the general reduction formula (3.61), the transition from $D_M\epsilon$ to $D_\mu\epsilon'$ introduces an extra sign $\epsilon_v = -1$. Comparing with (2.39), this can be matched by choosing $\epsilon_1 = -1$ in the corresponding expression of the three-dimensional model. With these two relations fixed, now for the more complicated terms.

The new gravitino kinetic term

The first of these is the recovery of the new Rarita Schwinger term from the reduced theory. From the Lagrangian of the three-dimensional model, (2.38), it is evident that this is the only term quadratic in the new gravitino. Thus, let us collect all such reduced terms, which must either be part of the ordinary Rarita-Schwinger term with three-dimensional Lorentz-covariant derivative, or part of the new connection coefficients $Q_\mu{}^{\bar{a}}{}_{\bar{b}}$ associated with the local $\mathfrak{so}(3)_B$-symmetry.

The quadratic terms in question come from two parts of the original five-dimensional Lagrangian (4.6). First of all, there is the original Rarita-Schwinger term. The general results in chapter 3 can be used to read off what it reduces to, namely the expression given in (3.55), involving the Hodge-dualized anholonomy coefficient $\Omega_{\alpha c}$

defined in (3.53). But there are further contributions in (4.6), namely those terms involving two gravitini Ψ^i_M and the field strength F_{MN}. For these, no general expression has been derived, making it necessary to perform the dimensional reduction explicitly. By now, the procedure is routine: A transition to flat indices where appropriate, the dimensional split including the case A split (4.9) of the gamma matrices, the redefinition (3.37) to arrive at the new gravitino Ψ'_μ and the rescaling $\tilde{\chi}_a := \Delta^{s/2}\Psi_a$ introduced in section 3.5, finished off with appropriate applications of gamma matrix algebra gives the result

$$\begin{aligned}
&\frac{\sqrt{3}}{4} i\kappa E \left[(\overline{\Psi}_{iA}\Psi^i_B)F^{AB} + \frac{1}{2}(\overline{\Psi}_{iA}\Gamma^{ABCD}\Psi^i_B)F_{CD} \right] \\
= \; &\frac{\sqrt{3}}{4} i\kappa e'\Delta^{-1} \Big\{ (\overline{\Psi}'_{i\mu}\gamma^{\mu\gamma\nu}[-\epsilon_3\hat{\Gamma}^v F_\gamma - \hat{\Gamma}^d F_{\gamma d}]\Psi'^i_\nu) + \frac{1}{2}\epsilon_v(\overline{\Psi}'_{i\mu}\gamma^{\mu\nu}\hat{\Gamma}^v\hat{\Gamma}^{cd}\Psi'^i_\nu)F_{cd} \\
&\quad + 2(\overline{\chi}_{ai}\gamma^\nu\gamma^\gamma[\epsilon_3\epsilon_v\hat{\Gamma}^a F_\gamma - \bar{\eta}^{ad}\hat{\Gamma}^v F_{\gamma d}]\Psi'^i_\nu) + 4(\overline{\chi}^c_i\hat{\Gamma}^d\gamma^\nu\Psi'^i_\nu)F_{cd} \\
&\quad + \epsilon_v(\overline{\chi}_{ai}[2(\hat{\Gamma}^a\bar{\eta}^{bd} - \hat{\Gamma}^d\bar{\eta}^{ab})F_{\gamma d} + \epsilon_3\hat{\Gamma}^v\hat{\Gamma}^{ab}F_\gamma]\gamma^\gamma\chi^i_b) \\
&\quad + (\overline{\chi}^c_i\hat{\Gamma}^v[6\hat{\Gamma}^d\hat{\Gamma}^b + \bar{\eta}^{db}]\chi^i_b)F_{cd} \Big\},
\end{aligned} \tag{4.25}$$

where the Hodge-dualization $F_\delta := 1/2 \cdot \varepsilon_\delta{}^{\alpha\beta}F_{\alpha\beta}$, introduced in (3.71), has been used. The terms quadratic in Ψ'_μ gleaned from this, combined with the contribution (3.55) from the original Rarita-Schwinger term, all ingredients for writing down the kinetic term for the new gravitino field are present. In the dimensionally reduced case, the result turns out to be

$$\begin{aligned}
&\frac{i}{2}e'(\overline{\Psi}'_{\mu i}\gamma^{\mu\nu\rho}\Big[D_\nu(\omega') - \tfrac{1}{4}e'_\nu{}^\alpha\{\hat{\Gamma}^{de}\Omega'_{\alpha de} + \epsilon_3\Delta\hat{\Gamma}^v\hat{\Gamma}^e\Omega'_{\alpha e} \\
&\qquad + 2\sqrt{3}\kappa\Delta^{-1}(\epsilon_3\hat{\Gamma}^v F_\alpha + \hat{\Gamma}^d F_{\alpha d})\}\Big]\Psi'^i_\rho)
\end{aligned} \tag{4.26}$$

with the abbreviation

$$D_\nu(\omega') := \partial_\nu + \tfrac{1}{4}\omega'_{\nu\alpha\beta}\gamma^{\alpha\beta} \tag{4.27}$$

for the new space-time covariant derivative. We have already seen in the general case how, in three dimensions, the additional terms bilinear in Ψ' can be interpreted as parts of a new covariant derivative. Equation (4.26) shows how this remains true even when the $F\Psi^2$-contributions are included.

Let us now revert to writing out explicitly the internal spinor indices; by the analysis of the new gravitino in section 4.2, they become $SO(3)_B$ indices $\bar{a}, \bar{b}, \bar{c}$ etc. Taking our cue from the enhanced covariant derivative

$$(D_\nu(\omega', Q)\chi)^{\bar{a}} = D_\nu(\omega')\chi^{\bar{a}} + Q_\nu{}^{\bar{a}}{}_{\bar{b}}\chi^{\bar{b}}, \tag{4.28}$$

defined for the three-dimensional theory in (2.22), we can identify from (4.26) the connection coefficients

$$\begin{aligned} Q_\nu{}^{\bar a}{}_{\bar b} \;:=\; & -\frac{1}{4} e'_\nu{}^{\alpha} \Big\{ (\hat\Gamma^{de})^{\bar a}{}_{\bar b} \Omega'_{\alpha de} + \epsilon_3 \Delta (\hat\Gamma^v \hat\Gamma^e)^{\bar a}{}_{\bar b} \Omega'_{\alpha e} \\ & + 2\sqrt{3}\kappa \Delta^{-1} \Big[\epsilon_3 (\hat\Gamma^v)^{\bar a}{}_{\bar b} F_\alpha + (\hat\Gamma^d)^{\bar a}{}_{\bar b} F_{\alpha d} \Big] \Big\}. \end{aligned} \tag{4.29}$$

Altogether, the gravitino kinetic term can be rewritten as

$$\mathcal{L}_{RSn} = \frac{i}{2} e' (\overline{\Psi}'_{\mu i \bar a} \gamma^{\mu\nu\rho} D_\nu(\omega', Q) \Psi'_\rho{}^{i\bar a}). \tag{4.30}$$

Taking into account the overall minus sign of the reduced Lagrangian, it is seen that this indeed matches the gravitino kinetic term of the three-dimensional theory given in (2.39). A quick check confirms that the coefficients $Q_\nu{}^{\bar a}{}_{\bar b}$ have the properties required of objects in the representation $(\mathbf{3}, \mathbf{1})$ of $SO(4) = SO(3)_B \times SO(3)_2$: They are traceless, and they obey the appropriate version $(Q_\nu{}^{\bar a}{}_{\bar b})^* = -Q_\nu{}^{\bar b}{}_{\bar a}$ of the symplectic reality condition, cf. (B.8).

Dimensional reductions as the one performed here lead to objects that are, relative to the enhanced symmetry, gauge-fixed. In the case of the $Q_\nu{}^{\bar a}{}_{\bar b}$, however, we can see directly in (4.29) how the anholonomy coefficients and field strength components that the dimensional reduction has yielded fit into the $SO(3)$. Remember that the $SO(3)$ gamma matrices were defined in terms of the internal gamma matrices as $\hat\Gamma^r = (\hat\Gamma^a, \hat\Gamma^v)$. For the field strength, it is possible to define an object $\bar F_{\beta r}$ that has both a space-time and an $SO(3)$ vector index by

$$\bar F_{\beta r} := \begin{cases} \bar F_{\beta c} & = F_{\beta c} \\ \bar F_{\beta v} & = \epsilon_3 F_\beta \end{cases} . \tag{4.31}$$

It occurs in (4.29) in the manifestly $SO(3)$-covariant form $\hat\Gamma^r \bar F_{\beta r}$. For the anholonomy coefficients, recall that the $SO(3)$-epsilon tensor was defined earlier by setting $\varepsilon^{vab} = -\varepsilon^{ab}$. With this definition, the components of objects V_r and T_{st} (T antisymmetric in s, t) contracted with ε^{rst} decompose into their (a, v) parts as

$$V_r \varepsilon^{rst} T_{st} = V_v \varepsilon^{vcd} T_{cd} + 2 V_a \varepsilon^{acv} T_{cv}.$$

Define an object with two antisymmetric $SO(3)$ vector indices and one space-time vector index by

$$\bar\Omega_{\alpha st} := \begin{cases} \bar\Omega_{\alpha cd} & = \Omega'_{\alpha[cd]} \\ \bar\Omega_{\alpha cv} & = -\bar\Omega_{\alpha vc} = 1/2 \cdot \Delta \epsilon_3 \Omega'_{\alpha c} \\ \bar\Omega_{\alpha vv} & = 0 \end{cases} . \tag{4.32}$$

The anholonomy coefficients occur in (4.29) in manifestly $SO(3)$-covariant form, to wit as $\hat\Gamma_r \varepsilon^{rst} \bar\Omega_{\alpha st}$.

The kinetic term of the matter fermion

The next goal is the kinetic term of the matter fermion. Its ingredients are the same as in the preceding section – the five-dimensional Rarita-Schwinger term and the five-dimensional $F\Psi^2$ terms. The appropriate parts of the dimensionally-split Rarita-Schwinger term have already been collected (3.58); the split of the $F\Psi^2$-term has been achieved in (4.25). This done, there are two redefinitions of the fermion fields to be inserted. The first is the simple rescaling, plus a linear combination involving matrices $\hat{\Gamma}$, given in (3.44). The second is the transition to the index structure appropriate for an object transforming under the **4** of the $SO(3)_2$, making use of (4.21) to pass from $\chi_a^{i\dot{a}}$ to $\chi^{i\dot{a}\dot{b}\dot{c}}$. The resulting expression can be drastically simplified using the identities (4.12)-(4.16) for the $\hat{\Gamma}$, with the – comparatively compact – result

$$\begin{aligned}
&\frac{i}{4}e'(\bar{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\nu D_\nu(\omega')\chi^{i\dot{a}\dot{b}\dot{c}})\\
&-\frac{3i}{16}e'(|\overline{c}|-1)(\bar{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\beta\chi^{i\dot{d}\dot{e}\dot{f}})\big[\Delta\epsilon_3(\hat{\Gamma}^v)^{\dot{g}}{}_{\dot{d}}\Omega'_{\beta c}+\tfrac{2}{\sqrt{3}}\kappa\Delta^{-1}\delta^{\dot{g}}_{\dot{d}}F_{\beta c}\big](\hat{\Gamma}^c)^{\dot{a}}{}_{\dot{g}}(\hat{\Gamma}^v)^{\dot{b}}{}_{\dot{e}}(\hat{\Gamma}^v)^{\dot{c}}{}_{\dot{f}}\\
&-\frac{3i}{16}e'(\bar{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\beta\big[\quad(\hat{\Gamma}^v)^{\dot{a}}{}_{\dot{d}}(\varepsilon^{cd}\Omega'_{\beta cd}+\tfrac{2}{\sqrt{3}}\epsilon_3\kappa\Delta^{-1}F_\beta)\\
&\qquad\qquad+(\overline{c}-2)(\hat{\Gamma}^c)^{\dot{a}}{}_{\dot{g}}(\Delta\epsilon_3(\hat{\Gamma}^v)^{\dot{g}}{}_{\dot{d}}\Omega'_{\beta c}-\tfrac{2}{\sqrt{3}}\kappa\Delta^{-1}\delta^{\dot{g}}_{\dot{d}}F_{\beta c})\big]\chi^{i\dot{d}\dot{b}\dot{c}}).
\end{aligned} \tag{4.33}$$

As can be seen from this expression, the objects attached to the $SO(3)_2$-indices of the matter fermions fall into three different representations. In the first line from the top, there is the trivial representation, as befits the action of the covariant derivative $D(\omega')$. In the third (and fourth) line, the number of indices and the tracelessness show that the corresponding object is in the **3** representation, suitable for an infinitesimal $SO(3)_2$ transformation and thus just as suitable for a connection coefficient. The last kind of object, that in the second line, is more problematic. By its index structure, it appears to belong to the **7**-representation of the $SO(3)_2$. It is not clear how to interpret such an object as part of a three-dimensional Lagrangian, and it certainly doesn't match the particular three-dimensional Lagrangian of interest here. Luckily, some freedom of choice for the constant $\overline{c}$ (which was used in the field redefinition (4.19)) remains: As seen from (4.23), the diagonality of $(\bar{\chi}_{i\dot{a}\dot{b}\dot{c}}\gamma^\nu D_\nu(\omega')\chi^{i\dot{a}\dot{b}\dot{c}})$ had forced $\overline{c}$ to take on one of the two values $+1$ and -2. From (4.33), it is evident that the choice $\overline{c}=1$ has the effect of getting rid of the disturbing term involving the objects in the **7**-representation. This fixes the last ambiguity in the redefinition of the matter fermions, and leaves an expression (4.33) that can be interpreted as a kinetic term with an enhanced covariant derivative. Recalling the action of the covariant derivative,

$$(D_\mu(\omega',Q)\chi)^{i\,\dot{a}\dot{b}\dot{c}} = D_\mu(\omega')\chi^{i\,\dot{a}\dot{b}\dot{c}} + 3\chi^{i\dot{d}(\dot{a}\dot{b}}Q_\mu{}^{\dot{c})}{}_{\dot{d}}$$

from (2.22), one can read off the connection coefficients

$$Q_\nu{}^{\dot a}{}_{\dot d} = -\frac{1}{4}e'_\nu{}^\beta\Big[(\hat\Gamma^v)^{\dot a}{}_{\dot d}(\varepsilon^{cd}\Omega'_{\beta cd} + \tfrac{2}{\sqrt3}\epsilon_3\kappa\Delta^{-1}F_\beta) - (\hat\Gamma^c)^{\dot a}{}_{\dot g}(\Delta\epsilon_3(\hat\Gamma^v)^{\dot g}{}_{\dot d}\Omega'_{\beta c} - \tfrac{2}{\sqrt3}\kappa\Delta^{-1}\delta^{\dot g}_{\dot d}F_{\beta c})\Big]. \quad (4.34)$$

Again, these are combinations of anholonomy coefficients, field strength components and internal gamma matrices that can be brought into manifestly $SO(3)$-covariant form, using objects like those defined in (4.32) and (4.31).

With the help of the enhanced covariant derivative $D(\omega', Q)$, the expression (4.34) can be brought into the very compact form

$$\frac{i}{4}(\bar\chi_{i\dot a\dot b\dot c}\gamma^\nu[D_\nu(\omega', Q)\chi]^{i\dot a\dot b\dot c}), \quad (4.35)$$

a standard kinetic term for spin-1/2 fermions. Remembering the overall sign difference between the reduced and the three-dimensional Lagrangian and recalling that $\epsilon_1 = -1$ was already fixed, there is an exact match with the three-dimensional model (2.38) as long as $\mu_1^2 = 2$. This determines the – hitherto free – parameter μ_1 (up to an overall sign). Let us choose $\mu_1 = \sqrt{2}$.

The Noether term and the determination of P_μ

This section is concerned with those terms in the dimensionally reduced Lagrangian that are to be combined into the three-dimensional Noether term, made up of those parts of the split Rarita-Schwinger and $F\Psi^2$-terms that contain one new gravitino and one new matter fermion. Once the Noether term is written down, comparison with (2.38) will give us the opportunity to read off the all-important derivative term P_μ of the scalars.

Once more, the first contribution comes from the kinetic term of the original gravitino field. It can be read off of the general expression (3.57) found in chapter 3. The $F\Psi^2$ term was already split a few pages back, and its contribution can be gleaned from eq. (4.25). Putting all this together, the Noether term starts out as

$$e'\frac{i}{2}(\overline{\chi}_{ai}[\tfrac12\Delta\epsilon_v(\epsilon_3)^{-1}(\hat\Gamma^a\hat\Gamma^c + \bar\eta^{ac})\Omega'_{\alpha c} - \hat\Gamma^v\hat\Gamma^c(\delta^a_c\Omega'_{\alpha d}{}^d + \bar\eta^{ab}\Omega'_{\alpha(bc)}) + \sqrt3\kappa\Delta^{-1}(\epsilon_3\epsilon_v\hat\Gamma^aF_\alpha - \bar\eta^{ad}\hat\Gamma^vF_{\alpha d})]\gamma^\rho\gamma^\alpha\Psi'^i_\rho). \quad (4.36)$$

Sure enough, the space-time gamma matrix structure $(\chi\gamma^\rho\gamma^\alpha\Psi_\rho)$ is exactly that needed for a match with the three-dimensional model. Then, it is again time for redefining the fermion fields: First the rescaling-plus-linear-combination of (3.44), with, as was fixed above, $\overline{c} = 1$, then the appropriate index structure $\chi^{i\dot a\dot b\dot c}$ via (4.21). Once more, heavy use of the identities (4.12)-(4.16) for the $\hat\Gamma$ is required, and once more the result is satisfyingly compact, namely

$$e'\frac{i}{4}(\overline{X}_{i\dot d\dot e\dot f}\gamma^\rho\gamma^\alpha\Psi'^{\,i\bar b}_\rho)\delta^{\dot b}_{\bar b}\Big\{\frac{1}{2}\Delta\epsilon_3\Omega'_{\alpha c}\left[\delta^{(\dot d}_{\dot b}(\hat\Gamma^c)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}+(\hat\Gamma^v\hat\Gamma^c)^{(\dot d}{}_{\dot b}(\hat\Gamma^v)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}\right]$$
$$+\left[(\hat\Gamma^v)^{(\dot d}{}_{\dot g}\delta^{\dot e}_{\dot b}\varepsilon^{\dot f)\dot g}\Omega'_{\alpha d}{}^d+(\hat\Gamma^v\hat\Gamma^c)^{(\dot d}{}_{\dot b}(\hat\Gamma^b)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}\Omega'_{\alpha(bc)}\right]$$
$$+\sqrt{3}\kappa\Delta^{-1}\left[F_{\alpha c}(\hat\Gamma^c)^{(\dot d}{}_{\dot g}(\hat\Gamma^v)^{\dot e}{}_{\dot b}\varepsilon^{\dot f)\dot g}+\epsilon_3F_\alpha(\hat\Gamma^v)^{(\dot d}{}_{\dot b}(\hat\Gamma^v)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}\right]\Big\}.$$

Comparing with the three-dimensional Lagrangian, (2.38), one can determine the gauge-fixed version of $P_\mu{}^{\bar a\dot a\dot b\dot c}$ defined by this expression. It is

$$(P_\mu)^{\bar a\dot d\dot e\dot f} = -\frac{i}{2\sqrt{2}}\,\varepsilon^{\bar a\dot b}e'_\mu{}^\alpha\Big\{\frac{1}{2}\Delta\epsilon_3\Omega'_{\alpha c}\left[\delta^{(\dot d}_{\dot b}(\hat\Gamma^c)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}+(\hat\Gamma^v\hat\Gamma^c)^{(\dot d}{}_{\dot b}(\hat\Gamma^v)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}\right]$$
$$+\left[(\hat\Gamma^v)^{(\dot d}{}_{\dot g}\delta^{\dot e}_{\dot b}\varepsilon^{\dot f)\dot g}\Omega'_{\alpha d}{}^d+(\hat\Gamma^v\hat\Gamma^c)^{(\dot d}{}_{\dot b}(\hat\Gamma^b)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}\Omega'_{\alpha(bc)}\right]$$
$$+\sqrt{3}\kappa\Delta^{-1}\Big[F_{\alpha c}(\hat\Gamma^c)^{(\dot d}{}_{\dot g}(\hat\Gamma^v)^{\dot e}{}_{\dot b}\varepsilon^{\dot f)\dot g}$$
$$+\epsilon_3F_\alpha(\hat\Gamma^v)^{(\dot d}{}_{\dot b}(\hat\Gamma^v)^{\dot e}{}_{\dot g}\varepsilon^{\dot f)\dot g}\Big]\Big\}.$$

By its index structure, this is manifestly an object in the required representation $(\mathbf{4},\mathbf{2})$ of the $SO(3)_2\times SO(3)_B$. It can be checked explicitly, using the reality conditions for the $\hat\Gamma$, that it satisfies the proper symplectic reality condition (2.20).

With P_μ defined in terms of the Ω' and F, all the entities P_μ, Q_μ are now fixed, and all the free parameters that needed to be determined in order to establish a correspondence between the dimensionally reduced and the three-dimensional model have been determined as well. With this out of the way, the remaining condition for an exact match of the Lagrangian and of the supersymmetry variations serve as consistency checks. They will be dealt with in the following section.

4.4 Matching the models II: Checks and Balances

Now for the checks and balances – the correspondences demanded by matching the dimensionally reduced to the three-dimensional model, at a stage where all parameters and all mappings of fields to fields are fixed. Some of the remaining matchings are well-nigh trivial (for instance, the dimensionally reduced vielbein had been defined in a way that adjusting $\lambda = i\kappa^2$ in (3.38) gives the proper correspondence in (2.39)). Some are not, and the following calculations will deal with three of these non-trivial checks:

The alternative derivation of ${Q_\nu}^{\bar{a}}{}_{\bar{b}}$ from the supersymmetry variation, which has to match the expression for ${Q_\nu}^{\bar{a}}{}_{\bar{b}}$ obtained from the new gravitino's kinetic term; the alternative derivation of $(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}}$ from the matter fermion's supersymmetry variation, which has to match the $(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}}$ derived from the Noether term; last, but far from least, the pièce de résistance: the match between the new kinetic term for the scalar fields, obtained by plugging $(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}}$ into the proper coset-model Lagrangian, and all bosonic terms of the reduced theory's Lagrangian – a match that requires the vector fields to be dualized to scalars, as is the custom in supergravities with hidden symmetries.

The gravitino supersymmetry variation

The $SO(3)_B$-connection coefficients ${Q_\nu}^{\bar{a}}{}_{\bar{b}}$ have already been derived from the Rarita-Schwinger kinetic term in the dimensionally reduced Lagrangian. Those same coefficients, part of the enhanced derivative, should occur in the new gravitino's supersymmetry variation. Let us start with the variation given in (4.7) and involving the supercovariant derivative (4.8). All the knowledge necessary to write down the dimensional split is contained in chapter 3. Eq. (3.62) shows the contributions from the five-dimensional Lorentz covariant derivative. The contributions of terms containing the field strength F, arising from the supercovariant derivative, can be read off (3.67), setting, appropriately, $d_2 = 2, b_2 = 4, s = -1$ and $b_1 = \kappa/4\sqrt{3}$. Once more, the next step is to dualize $F_{\alpha\beta}$, as given in (3.71). The result is

$$\delta_S \Psi'^i_\mu = -\Big\{\partial_\nu + \tfrac{1}{4}\omega'_{\nu\alpha\beta}\gamma^{\alpha\beta} - \tfrac{1}{4}e'_\nu{}^\alpha\Big[\hat{\Gamma}^{de}\Omega'_{\alpha de} + \epsilon_3\Delta\hat{\Gamma}^v\hat{\Gamma}^e\Omega'_{\alpha e} + 2\sqrt{3}\Delta^{-1}\epsilon_\delta\kappa\big(\epsilon_3\hat{\Gamma}^v F_\delta + \hat{\Gamma}^b F_{\alpha b}\big)\Big]\Big\}\epsilon'^i \tag{4.37}$$

which is exactly

$$\delta_S \Psi'^{i\dot{a}}_\mu = -D_\mu(\omega', Q)\epsilon'^{i\dot{a}}, \tag{4.38}$$

for the $SO(3)$-covariant derivative and connection coefficients defined in (4.28) and (4.29), respectively. Keeping in mind that, above $\epsilon_1 = -1$ has already been fixed, this is the variation given in (2.39).

Supersymmetry variation of the matter fermions χ

The next consistency check concerns the supersymmetry variation of the matter fermion which, as per eq. (2.39), can be used to read off P_μ. Once more, chapter 3 with its general formulae serves as a starting point: With eq. (3.69) for the splitting of the Lorentz covariant derivative, and (3.70) for the field-strength dependent part of the supercovariant derivative, one can write down the new matter fermion's variation explicitly. Once again, what follows is an application of the redefinition procedure to arrive at the new fermions $\chi^{i\dot{a}\dot{b}\dot{c}}$, namely eq. (3.44) with $\bar{c} = 1$, and eq. (4.21), relying on the identities (4.12)-(4.16) for the $\hat{\Gamma}$ for simplification. In comparison with (2.39), the result enables us to read off $(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}}$. Consistency is achieved if

$$\frac{1}{4\mu_1} = \frac{\mu_1}{8},$$

in other words, if $\mu_1^2 = 2$, which is the choice of parameter μ_1 that was already found necessary following eq. (4.35).

The dualized bosonic Lagrangian

As a final check, this section will deal with the construction of the dimensionally reduced bosonic Lagrangian. This ties together the previous matchings, all of which involved fermionic parts of the theory, with the bosonic side: On the one hand, it is possible to obtain the bosonic Lagrangian directly, by dimensional reduction of the five-dimensional bosonic Lagrangian, but it can also be determined by inserting $(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}$, which has been found to be the expression given in eq. (4.37), into the proper kinetic term of the three-dimensional coset model. As is usual in the construction of hidden symmetries, the match can only be achieved if the space-time vector fields that have arisen from the dimensional reduction are dualized to scalars.

First, for the kinetic term derived from $(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}$. Inserting $(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}$ given in (4.37) into the kinetic term given in eq. (2.38), and using identities like the Clifford algebra relations for the matrices $\hat{\Gamma}^a$, the resulting expression for the bosonic Lagrangian is

$$\begin{aligned}
\frac{i}{\kappa^2} e'(P_\mu)_{\bar{a}\dot{a}\dot{b}\dot{c}}(P^\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}} &= e'\Big\{ \frac{1}{4\kappa^2}[\Omega'_{\alpha a}{}^a \Omega'^\alpha{}_b{}^b + \Omega'_{\alpha(bc)}\Omega'^{\alpha(bc)}] - \frac{1}{8\kappa^2}\Delta^2 \Omega'_{\alpha c}\Omega'^{\alpha c} \\
&\qquad - \frac{1}{2}\Delta^{-2}F_{\alpha d}F^{\alpha d} + \frac{1}{2}\Delta^{-2}F_\alpha F^\alpha \Big\} \\
&= e'\Big\{ -\frac{1}{16\kappa^2}(\partial_\nu \bar{g}_{mn})(\partial^\nu \bar{g}^{mn}) + \frac{1}{4\kappa^2}(\partial_\nu \ln\Delta)(\partial^\nu \ln\Delta) \\
&\qquad + \frac{1}{16\kappa^2}\Delta^2 \bar{g}_{mn} G^m_{\mu\nu} G^{\mu\nu n} \\
&\qquad - \frac{1}{2}(\partial_\mu A_m)(\partial^\mu A_n)\bar{g}^{mn} - \frac{1}{4}\Delta^{-2}F'_{\mu\nu}F'^{\mu\nu} \Big\}, \qquad (4.39)
\end{aligned}$$

which is innocent enough – it contains clearly recognizable kinetic terms for the space-time scalars Δ, $\bar{g}_{mn}$ and A_m, and, extra "dilaton" factors Δ aside, Maxwell-like kinetic terms involving the field strengths $G^m_{\mu\nu}$ and $F'_{\mu\nu}$ of $B_\mu{}^m$ and A'_μ, respectively (although, as seen in eq. (3.21), the relationship between A'_μ and $F'_{\mu\nu}$ is a bit less straightforward than in ordinary Maxwell theory).

Next, for the dimensional reduction on the bosonic part of the Lagrangian: the Einstein-Hilbert, the Maxwell and the Chern-Simons term A^2F. The groundwork for this has been laid in section 3.2: For the Einstein-Hilbert term, the expression (3.18) can be used; for the Maxwell term, the expression (3.22). For the Chern-Simons term, a dimensional split, followed by throwing away the terms F_{ab} that vanish in the case of dimensional reduction, leads to

$$e'\Delta^{-2}\left\{\varepsilon^{\alpha\beta\delta}\varepsilon^{ce}A_\alpha F_{\beta c}F_{\delta e}+\varepsilon^{\beta\gamma\delta}\varepsilon^{ae}A_a F_{\beta\gamma}F_{\delta e}-\frac{1}{2}\varepsilon^{\alpha\beta\gamma}\varepsilon^{de}A_\alpha F_{\beta\gamma}F_{de}\right\}\mathrm{d}^5x. \quad (4.40)$$

It is convenient to perform one partial integration to eliminate the lone A_α; the result of this is

$$\frac{1}{2}e'\varepsilon^{\beta\gamma\delta}\varepsilon^{ae}A_aF_{\delta e}[3\Delta^{-2}F_{\beta\gamma}-e'_\beta{}^\mu e'_\gamma{}^\nu G^p_{\mu\nu}A_p]. \quad (4.41)$$

All in all, the dimensionally reduced bosonic Lagrangian now reads

$$\begin{aligned}\frac{1}{4\kappa^2}e'\mathcal{R}' \quad + \quad e'\Big\{&\frac{1}{16\kappa^2}\Delta^2\bar{g}_{mn}G^m_{\mu\nu}G^{\mu\nu n}+\frac{1}{16\kappa^2}(\partial^\mu\bar{g}_{mn})(\partial_\mu\bar{g}^{mn})\\ &-\frac{1}{4\kappa^2}(\partial^\mu\ln\Delta)(\partial_\mu\ln\Delta)\\ &-\frac{1}{4}\Delta^{-2}F'_{\mu\nu}F'^{\mu\nu}+\frac{1}{2}\bar{g}^{mn}(\partial_\mu A_m)(\partial_\nu A_n)\\ &-\epsilon_3\kappa\frac{1}{3\sqrt{3}}\,\varepsilon^{\mu\nu\rho}\varepsilon^{mn}A_m(\partial_\rho A_n)[3F'_{\mu\nu}+\Delta^2G^p_{\mu\nu}A_p]\Big\}. \quad (4.42)\end{aligned}$$

Comparing with (4.39), and keeping in mind the overall sign difference between the reduced Lagrangian and its three-dimensional counterpart, one finds the following: The kinetic terms for the scalar fields A_m and $e_m{}^a$ (the latter represented by $\bar{g}_{mn}$ and Δ) are in perfect agreement. For the remaining terms, all of them involving the vector field strengths $F'_{\mu\nu}$ and/or $G^m_{\mu\nu}$, there is no agreement – yet. The apparent disagreement is, however, not surprising. As in all other models with hidden symmetry, the coset Lagrangian can only be constructed out of space-time scalars. The direct dimensional split of the bosonic field leaves us with some of the scalars needed (Δ, $e_m{}^a$ and A_m), but also with space-time vectors A_μ and $B_\mu{}^m$. In three-dimensional space-time there is, at least on the classical level, a Hodge-dual description of these vector fields in terms of scalars: There, the field strength F corresponding to a vector (or one-form) A is a two-form, and the Bianchi-identity is hence a three-form. Resorting to a first-order formalism, it is possible to enforce the Bianchi identity by adding a corresponding constraint term to the Lagrangian, introducing a Lagrange multiplier field in the process. As the Bianchi identity is a three-form already, that Lagrange multiplier must be a scalar ϕ. Turning the tables, one partial integration turns the constraint term into a term containing the field strength and the Lagrange multiplier's derivative (or "field strength") $\mathrm{d}\phi$, and by solving the equation of motion for the field strength F, one obtains an equation of motion linking $\mathrm{d}\phi$ with the fields A and F. If, as usually the case for gauge fields, A does not occur explicitly in the Lagrangian, this equation is purely algebraic and readily solved. In the end, one can revert to a second order formalism, but now with the dual field ϕ as the only remaining field – Bianchi identity and field equation have exchanged their roles, and at least on the classical level, the new Lagrangian involving only a scalar field describes the same physics as the old Lagrangian with the vector field A did.

This is the dualization procedure that will now be applied to the vector fields. The first dualized-field-to-be is $G^m_{\mu\nu} = 2\partial_{[\mu}B^m_{\nu]}$, and it is obvious from its definition that it satisfies the Bianchi identity $\partial_{[\mu}G^m_{\nu\rho]} = 0$. However, with A'_μ, more care is needed: $F'_{\mu\nu}$ does not satisfy such a Bianchi identity; still, it is readily verified that the modified mixed field strength

$$\tilde{F}_{\mu\nu} := \Delta^{-2}F'_{\mu\nu} - G^m_{\mu\nu}A_m \tag{4.43}$$

does satisfy $\partial_{[\mu}\tilde{F}_{\nu\rho} = 0$, as required. Furthermore, it can be seen from (4.42) that for these fields and their corresponding vector fields A'_μ and ${B_\mu}^m$, only the (gauge invariant) field strength occurs in the Lagrangian – which bodes well for the intended dualization.

For each Bianchi identity, introduce a Lagrange multiplier field: φ for the identity $\partial_{[\mu}\tilde{F}_{\nu\rho} = 0$, and, with an additional internal curved index, ξ^n for $\partial_{[\mu}G^m_{\nu\rho]} = 0$. The next step is to add to the original Lagrangian (4.42) the two constraint terms

$$-\frac{1}{2}e'\,\varepsilon^{\mu\nu\rho}\phi(\partial_\mu\tilde{F}_{\nu\rho}) - \frac{1}{2}e'\,\varepsilon^{\mu\nu\rho}\xi_m(\partial_\mu G^m_{\nu\rho}). \tag{4.44}$$

In the new Lagrangian, we can vary with respect to $G^m_{\mu\nu}$ and with respect to $F'_{\mu\nu}$ in order to obtain the equations of motion for these fields. They can be disentangled by linear combination, giving the two simple equations

$$G^m_{\mu\nu} = 4\kappa^2\,{\varepsilon_{\mu\nu}}^\rho\left\{\Delta^{-2}(\partial_\rho\phi)A^m - \Delta^{-2}(\partial_\rho\xi_n)\bar{g}^{mn} + \frac{2}{3\sqrt{3}}\epsilon_3\kappa A^m\varepsilon^{nr}A_n(\partial_\rho A_r)\right\}$$

and

$$F'_{\mu\nu} = {\varepsilon_{\mu\nu}}^\rho\left\{(\partial_\rho\varphi) - \frac{2}{\sqrt{3}}\epsilon_3\Delta^2\kappa\varepsilon^{nr}A_n(\partial_\rho A_r)\right\} \tag{4.45}$$

that express each of the two field strengths purely in terms of the new and old scalars. This can be substituted back into the new Lagrangian formed by the two kinetic terms for $G^m_{\mu\nu}$ and $F'_{\mu\nu}$, the Chern-Simons term and the additional constraint terms (4.44), with the result

$$\begin{aligned}
&-\frac{1}{2}\Delta^{-2}g'^{\rho\lambda}\left((\partial_\rho\varphi) - \frac{2}{\sqrt{3}}\epsilon_3\Delta^2\kappa\varepsilon^{nr}A_n(\partial_\rho A_r)\right)\cdot\left((\partial_\lambda\varphi) - \frac{2}{\sqrt{3}}\epsilon_3\Delta^2\kappa\varepsilon^{nr}A_n(\partial_\lambda A_r)\right)\\
&+2\kappa^2\Delta^{-2}g'^{\rho\lambda}\left((\partial_\rho\varphi)A^m - (\partial_\rho\xi_p)\bar{g}^{mp} + \frac{2}{3\sqrt{3}}\epsilon_3\kappa A^m\varepsilon^{pr}A_p(\partial_\rho A_r)\right)\\
&\qquad\cdot\left((\partial_\rho\varphi)A_m - (\partial_\rho\xi_m) \quad + \frac{2}{3\sqrt{3}}\epsilon_3\kappa A_m\varepsilon^{pr}A_p(\partial_\rho A_r)\right).
\end{aligned} \tag{4.46}$$

This is exactly the same result as if one had substituted the two dualization equations (4.45) and (4.45) directly into the kinetic term for the coset objects $(P_\mu)^{\bar{a}\dot{a}\dot{b}\dot{c}}$ in eq. (4.39).

The final check completes the derivation: we have successfully uncovered the hidden symmetry in the $d = 5, \mathcal{N} = 2$ supergravity, reducing the model to $d = 3$ and matching it up with the three-dimensional $G_{2(+2)}/SO(4)$ coset supergravity.

Chapter 5

From supercharges to quaternionic spinors

With the matchings of the previous chapters, the recovery of the hidden symmetries is complete, and the reader interested only in those results may well wish to jump forward to the summary and outlook presented in chapter 6. This chapter is (in the everyday as well as in the mathematical sense of the word) a spin-off from the results presented in the preceding chapters. It investigates, in a more general setting, some of the mathematical structures occurring in the analysis of supercharges that we have encountered in section 2.2.

There, the examination of the group representations involved in the construction of the three-dimensional model had commenced with a study of the supercharges Q and the R-symmetry associated with them. This gave an indication of how to implement the local symmetry H in the model under study; in other applications, such an analysis might be indicated to determine H in the first place. The involvement of Majorana conditions leads to the fact that such an analysis is possible on a rather abstract level, namely in terms of spin-invariant automorphism groups of real Clifford algebras[1]. As is usual when dealing with real Clifford algebras, the associated representation matrices are, in turns, real, complex or quaternionic. In the case examined here, for instance, the splitting of gamma matrices corresponds to a quaternionic Clifford algebra isomorphic to the matrix algebra[2] $\mathbb{H}(2)$, and it decomposes into the space-time Clifford algebra $\mathbb{R}(2)$ and the internal Clifford algebra $\mathbb{H}$, the latter giving rise to the enhanced local symmetry $U(1,\mathbb{H}) \simeq USp(2,\mathbb{C}) \simeq SU(2,\mathbb{C})$. (However, the results of section 2.2 show the limitations of this analysis – in the present case, only half the local symmetry can be obtained this way, and the analysis of the gravitino does not give the $SO(3)_2$.) This chapter grew out of the question in how far the quaternionic structures that occur in the abstract analysis can be made visible in the spinor part of

[1]To the best of my knowledge, the first appearance of this explicit relation between the real Clifford algebras and the local enhanced symmetries is in [27], dealing with the eleven-dimensional supergravity. For a review of supergravity in which the R-symmetries are analyzed in this way, see [38].

[2]In the following, the algebra of $n \times n$ matrices over a division algebra $\mathbb{D}$ will be denoted by $\mathbb{D}(n)$.

the field theory[3]. Explicit quaternionic representations of space-time spinors are not often seen in physics[4], but they do occur, chiefly in models that construct not only spinors, but all fields (and objects like derivatives) in terms of quaternions, and often restricted to a specified space-time dimension [99, 100, 91, 1, 16, 17, 49]. The aim of this chapter is to supplement these works by a systematic analysis starting from the abstract properties of real Clifford algebras, ending with a "minimal formalism" to write spinors in quaternionic form. On the way, I will construct tabular representations of all real Clifford algebras isomorphic to quaternionic matrix algebras, and obtain some results about quaternionic Fierz identities that complement the analysis in [113].

For a start, let us review the well-known basic properties of real Clifford algebras [6, 117, 8, 89, 27, 116]. Define the real Clifford algebra $\mathcal{C}(p,q)$ as the real free algebra generated by elements $e_i, i = 1, \ldots n$, $n = p + q$, with the additional relation

$$\{e_i, e_j\} = \eta_{ij}\mathbb{1}, \tag{5.1}$$

with η the generalized Minkowski metric with signature p, q and $\mathbb{1}$ the unit. Furthermore, demand the non-degeneracy condition $e_1 e_2 \cdots e_{p+q} \neq \mathbb{1}$. Then, $\mathcal{C}(p,q)$ is spanned by ordered products of the generators, i.e. all elements of the form $e_{i_1} e_{i_2} \cdots e_{i_k}$ with $i_1 < i_2 < \ldots < i_k$. Introducing the notation $e_{i_1 i_2 \ldots i_k} := e_{i_1} e_{i_2} \cdots e_{i_k}$, the aforementioned generators will be called the canonical basis of $\mathcal{C}(p,q)$, and the maximal basis element $\epsilon := e_{12\ldots n}$ the "volume element". The linear independence of the canonical basis elements follows from the Clifford relation (5.1) [120, Section 1.4]. Next, introduce two canonical anti-involutions: Clifford reversion $a \mapsto \tilde{a}$ reverses the order of generators in each basis element; Clifford conjugation $a \mapsto a^+$ does the same and adds a minus sign for each generator. In the following, it proves useful to introduce an additional, non-standard anti-involution I will call *Clifford transposition*. It is defined as reversion combined with mapping each single generator to its inverse. Given this definition, it is possible to lift the concept of the *generalized charge conjugation operator* of matrix representations [112] to the abstract algebra. Define the charge conjugation element C as $C = e_1 \cdots e_p$ if p is even, and as $C = e_{p+1} \cdots e_{p+q}$ for odd p. Then, for all $a \in \mathcal{C}(p,q)$,

$$a^T = \begin{cases} C^{-1}\, \tilde{a}\, C & p \text{ and } n \text{ odd} \\ C^{-1}\, a^+ C & \text{all other cases;} \end{cases} \tag{5.2}$$

for the generators e_i, this follows from directly inspecting the different possibilities arising from $i < p$ vs. $i > p$ and p even vs. p odd; for products of generators, the implicit reversion ensures that the generators involved cancel each other without the

[3]It is a much more involved question in how far these quaternionic structures are visible on the bosonic side, in the coset model - this leads from the R-symmetries to the study of special geometries associated with those non-linear sigma models allowed by the supersymmetry [43, 45, 44].

[4]This is somewhat ironic, given the fact that the very first construction of the Clifford algebras, whose complex and Majorana incarnations have found so wide-spread use in physics, did in fact centre on the quaternions [23].

need for reordering. The importance of this construction is the following: Up to equivalence, Clifford conjugation and reversion, viewed as adjoints of a scalar product, are sufficient to construct all Spin$^+$-invariant products [8, Section 2.6]. In the construction of tabular representations, below, we will encounter standard isomorphisms, relating Clifford algebras to tensor products of certain other Clifford algebras. They respect Clifford transposition in the sense that transposition on the product induces transposition on the factors; the same is not generally true for conjugation or reversion.

With this anti-involution established, back to results from the standard literature: Trivially, the unit element of the algebra is invariant under algebra automorphisms; this makes the scalar projection of a general algebra element to the unit, practically "setting all generators e_i zero", namely $a = a_0\mathbb{1} + a_1 e_1 + a_2 e_2 \cdots \mapsto (a)_S = a_0$, a useful concept. With this projection and using the algebra relation, one can define projections onto each of the canonical basis elements, and thus explicitly decompose an algebra element in relation to that basis. Introducing indices raised by the metric η, this can be compactly rewritten as

$$a = \sum_{k=0}^{n} \left[\frac{1}{k!} (e^{i_k \ldots i_1} a)_S \; e_{i_1 \ldots i_k} \right], \tag{5.3}$$

the abstract algebra version of a Fierz identity.

There are several conceptually different ways of examining the representation theory of real Clifford algebras [50, 10, 90, 112]. Here, I will follow the classical constructive approach where higher-dimensional representations are systematically built from lower-dimensional ones[5]. Three types of ingredients are used – first of all, isomorphisms between a Clifford algebra and a (real) tensor product of Clifford algebras[6]

$$\begin{aligned} \mathcal{C}(p+1,q+1) &\simeq \mathcal{C}(p,q)\otimes\mathcal{C}(1,1) \\ \mathcal{C}(q,p+2) &\simeq \mathcal{C}(p,q)\otimes\mathcal{C}(0,2) \\ \mathcal{C}(q+2,p) &\simeq \mathcal{C}(p,q)\otimes\mathcal{C}(2,0) \\ p\geq 1: \quad \mathcal{C}(p,q) &\simeq \mathcal{C}(q+1,p-1). \end{aligned} \tag{5.4}$$

The second type of isomorphisms concerns the real associative division algebras $\mathbb{D} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ and their total matrix algebras, namely[7] $\mathbb{R}(n)\otimes\mathbb{D}(m) = \mathbb{D}(nm)$, $\mathbb{H}\otimes\mathbb{H}\simeq\mathbb{R}(4)$ and $\mathbb{H}\otimes\mathbb{C}\simeq\mathbb{C}(2)$. Finally, one can explicitly construct representations of certain lower-dimensional Clifford algebras: $\mathcal{C}(1,0)\simeq\mathbb{R}\oplus\mathbb{R}$, noting that $(1\pm e_1)/2$

[5]See [6, Part I, §4] or, for a more general treatment, [88].

[6]The first three can be proven directly by noting that, for e_i ($i = 1,\ldots n$) the generators of $\mathcal{C}(p,q)$ and $\bar{e}_j$ ($j = 1, 2$) the generators of $\mathcal{C}(1,1)$, $\mathcal{C}(2,0)$ or $\mathcal{C}(0,2)$, the elements $e_i\otimes\bar{e}_1\bar{e}_2$ ($i = 1,\ldots,n$) and $\mathbb{1}\otimes\bar{e}_j$ ($j = 1,2$) of the tensor product algebra generate the appropriate Clifford algebra (this is, in fact, the reverse of the dimensional splitting procedure that was discussed in section 3.3). For the last isomorphism, starting with $\mathcal{C}(p,q)$, the map $e_1\mapsto e_1$, and, for $2\leq i\leq n$, $e_i\mapsto e_1 e_i$ is an algebra isomorphism onto $\mathcal{C}(q+1,p-1)$.

[7]E.g. [8, App. A]

projects unto the components. $\mathcal{C}(0,1) \simeq \mathbb{C}$, with $e_1 \simeq i$. $\mathcal{C}(0,2) \simeq \mathbb{H}$, where $e_1 \simeq i, e_2 \simeq j$. Last but not least, $\mathcal{C}(1,1) \simeq \mathbb{R}(2)$, which can be seen defining the *real Pauli matrices*, which are related to their more common counterparts (A.17) by setting σ_2[real] $= -i\sigma_2$[usual]. These matrices can be mapped to the basis elements of $\mathcal{C}(1,1)$ as $\sigma_0 \mapsto \mathbb{1}, \sigma_1 \mapsto e_1, \sigma_2 \mapsto e_2, \sigma_3 \mapsto e_{12}$. From this, the Clifford algebras $\mathcal{C}(p,0)$ and $\mathcal{C}(0,p)$ can be constructed for $p = 0, \ldots 8$. With repeated application of (5.4), a general $\mathcal{C}(p,q)$ can be decomposed into either $\mathcal{C}(p-q,0) \otimes \mathbb{R}(2^q)$ or $\mathcal{C}(0,q-p) \otimes \mathbb{R}(2^p)$. All in all, this leads to the isomorphisms

$(p-q)$ mod 8	$\mathcal{C}(p,q) \simeq$
$0,2$	$\mathbb{R}(2^{n/2})$
1	$\mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]})$
$3,7$	$\mathbb{C}(2^{[n/2]})$
$4,6$	$\mathbb{H}(2^{n/2-1})$
5	$\mathbb{H}(2^{[n/2]-1}) \oplus \mathbb{H}(2^{[n/2]-1})$

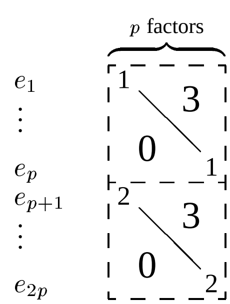
(5.5)

The result $\mathcal{C}(8,0) \simeq \mathbb{R}(16) \simeq \mathcal{C}(0,8)$ and the isomorphisms introduced above lead to the famous (Bott-)periodicity property

$$\mathcal{C}(p+8l, q+8m) \simeq \mathcal{C}(p,q) \otimes \mathbb{R}(16^{l+m}).$$

In the same way as this classification, explicit matrix representations can be constructed. The underlying structure and periodicity allows the construction of "tabular representations", schemes from which a given representation can be read right off, without the need to go explicitly through the tensorings and isomorphisms. Some of these schemes can be found in the literature – the real matrix algebras are tabulated in the appendix of [7], and general tables with $\mathbb{C}$ and $\mathbb{H}$ expressed as subalgebras of $\mathbb{R}(2)$ and $\mathbb{R}(4)$, respectively, in [128]. For the purposes of this analysis, tabular representations for the quaternionic matrix algebras are needed and, in preparation, also tabular representations for selected real matrix algebras.

A suitable starting point is the case $\mathcal{C}(p,p)$. Repeated tensoring of $\mathcal{C}(1,1)$ with itself on the Clifford algebra side, and the same for $\mathbb{R}(2)$ on the representation side, gives the representation shown below, where the notation is the following: Each line contains one of the generators e_i as well as its representative; for better legibility a short-hand notation is employed: inside the dashed box, the numbers indicate a tensor product of real Pauli matrices, e.g., $0\ 3\ 2$ means $\sigma_0 \otimes \sigma_3 \otimes \sigma_2$.



(5.6)

The general expression for $\mathcal{C}(p,q)$ with $p-q=0 \mod 8$ can be obtained by repeated application of one of the block substitutions

$$\begin{array}{cccc} 2&3&3&3\\ 0&2&3&3\\ 0&0&2&3\\ 0&0&0&2 \end{array} \mapsto \begin{array}{cccc} -0&2&1&2\\ 2&3&1&2\\ -2&1&0&2\\ 2&1&2&3 \end{array} =: A_+ \quad \text{or} \quad \begin{array}{cccc} 1&3&3&3\\ 0&1&3&3\\ 0&0&1&3\\ 0&0&0&1 \end{array} \mapsto \begin{array}{cccc} 0&1&2&1\\ -1&3&2&1\\ 1&2&0&1\\ -1&2&1&3 \end{array} =: A_-$$

to the block of tensored Pauli matrices, with the leading 0s and the trailing 3s unchanged. They are the matrix version of the isomorphism $\mathcal{C}(p,q) \simeq \mathcal{C}(p-4,q+4)$, valid for $p \geq 4$ and effected by mapping $e_i \mapsto e_1e_2e_3e_4e_i$ for $i=1,\ldots,4$ while leaving the rest of the generators unchanged; substitutions of this kind yield matrix expressions for the generators of $\mathcal{C}(k+8l, k+8m)$.

Next, for the case of $p-q \mod 8 = 2$. For $\mathcal{C}(2,0)$, there is a representation $e_1 \simeq \sigma_3, e_2 \simeq \sigma_1$; tensoring k times with $\mathcal{C}(1,1)$ leads to the representation for $\mathcal{C}(2+k,k)$ generated by

$$\begin{array}{l|cccc|} & \multicolumn{4}{c}{\overbrace{\qquad\qquad}^{k+1 \text{ factors}}} \\ e_1 & 1 & 3 & & \\ e_2 & 3 & 3 & 3 & \\ e_{2+1} & 0 & 1 & & \\ \vdots & & & \ddots & \\ e_{2+k} & & 0 & & 1 \\ \hline e_{2+k+1} & 0 & 2 & 3 & \\ \vdots & & & \ddots & \\ e_{2+2k} & & 0 & & 2 \end{array} \tag{5.7}$$

and block substitutions using A_+ and A_- given above readily lead to the generators of $\mathcal{C}(2+k+8l, k+8m)$. The enlargement procedure is of mechanical simplicity – write the representatives of the original generators; fill everything directly below them with the appropriate number of zeroes and everything behind with 3s; fit in the two simple blocks (5.6); apply $A_\pm$ as needed. Given their simplicity, I will not explicitly spell these steps out in the following, but concentrate on those starting points for such enlargement that involve the quaternions. For $p-q \mod 8 = 6$ and $q>p$, we already know how $\mathcal{C}(0,2) \simeq \mathbb{H}$. For $p>q$, use $\mathcal{C}(6,0) \simeq \mathcal{C}(2,4) \simeq \mathcal{C}(0,1)[\otimes\mathcal{C}(1,1)]^2$, following which one can construct

$$\begin{array}{cr|cc|} e_1 & -j & 1 & 2\\ e_2 & i & 1 & 2\\ e_3 & k & 0 & 2\\ e_4 & -k & 2 & 3\\ e_5 & 1 & 1 & 3\\ e_6 & 1 & 0 & 1 \end{array} \tag{5.8}$$

For $p-q \mod 8 = 4$ and $p>q$, one can use $\mathcal{C}(4,0) \simeq \mathcal{C}(2,0) \otimes \mathcal{C}(0,2)$, giving

$$e_1 \simeq i\sigma_2, \quad e_2 \simeq j\sigma_2, \quad e_3 \simeq \sigma_1, \quad \simeq \sigma_3 \tag{5.9}$$

while commutativity of the tensor product, applied to $\mathcal{C}(2,0) \otimes \mathcal{C}(0,2)$ yields the starting point for the case $q>p$, namely

$$e_1 \simeq j\sigma_0, \quad e_2 \simeq -i\sigma_0, \quad e_3 \simeq -k\sigma_3, \quad e_4 \simeq k\sigma_1. \tag{5.10}$$

Finally, for the case $p - q \mod 8 = 5$. The simpler case is $q > p$, where it is possible to use (5.4), namely $\mathcal{C}(0,3) \simeq \mathcal{C}(1,0) \otimes \mathcal{C}(0,2)$ to construct

$$e_1 \simeq -k \oplus k, \quad e_2 \simeq i \oplus i, \quad e_3 \simeq j \oplus j. \tag{5.11}$$

For $p > q$, I use $\mathcal{C}(5,0) \simeq \mathcal{C}(1,0) \otimes \mathcal{C}(4,0)$ as well as the representation that was found for $\mathcal{C}(4,0)$ in eq. (5.9), above, to construct

$$\begin{array}{ll} e_1 & (\;k \oplus -k) \\ e_2 & (\;i \oplus \;\;i\,) \\ e_3 & (\;j \oplus \;\;j\,) \\ e_4 & (\;1 \oplus \;\;1\,) \\ e_5 & (\;1 \oplus \;\;1\,) \end{array} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \tag{5.12}$$

It is a crucial property of these representations (as well as analogous representations for the other Clifford algebras, involving real and complex matrices) that Clifford transposition corresponds to quaternionic Hermitian conjugation (in the real case, transposition; in the complex, Hermitian conjugation). This will allow us to implement spin-invariant products in the usual physicist's way, by Dirac conjugation. Such products have as their adjoint either Clifford reversion or Clifford conjugation, and the determination of their properties proceeds in the same way as the classification, working in parallel with isomorphisms on the Clifford algebra and on the representation side. The results[8] for those quaternionic Clifford algebras of possible interest in physics (i.e. belonging to spaces with at most one time dimension) are contained in the following table.

$p-q$	$q=0$		$q=1$		$p=0$		$p=1$	
mod 8	REV	C-C	REV	C-C	REV	C-C	REV	C-C
4	$\bar{\mathbb{H}}$–sym (2,0) $\mathbb{1}$	$\bar{\mathbb{H}}$–sym (1,1) ϵ	$\bar{\mathbb{H}}$–sym (2,2) $e_n\epsilon$	$\bar{\mathbb{H}}$–skew (4) e_n	$\bar{\mathbb{H}}$–sym (1,1) ϵ	$\bar{\mathbb{H}}$–sym (2,0) $\mathbb{1}$	$\bar{\mathbb{H}}$–sym (2,2) e_1	$\bar{\mathbb{H}}$–skew (4) $e_1\epsilon$
5	${}^2\bar{\mathbb{H}}$–sym (2,0) $\mathbb{1}$	${}^2\mathbb{H}$–swap (2) $\emptyset$	${}^2\mathbb{H}$–swap (4) $\emptyset$	${}^2\bar{\mathbb{H}}$–skew (4) e_n	${}^2\mathbb{H}$–swap (1) $\emptyset$	${}^2\bar{\mathbb{H}}$–sym (1,0) $\mathbb{1}$	${}^2\bar{\mathbb{H}}$–sym (1,1) e_1	${}^2\mathbb{H}$–swap (2) $\emptyset$
6	$\bar{\mathbb{H}}$–sym (4,0) $\mathbb{1}$	$\bar{\mathbb{H}}$–skew (4) ϵ	$\bar{\mathbb{H}}$–skew (8) $e_n\epsilon$	$\bar{\mathbb{H}}$–skew (8) e_n	$\bar{\mathbb{H}}$–skew (1) ϵ	$\bar{\mathbb{H}}$–sym (1,0) $\mathbb{1}$	$\bar{\mathbb{H}}$–sym (1,1) e_1	$\bar{\mathbb{H}}$–sym (1,1) $e_1\epsilon$

In this table, p and q define the Clifford algebra in question, while REV and C-C specify the scalar product's adjoint as reversion or Clifford conjugation, respectively. Each field contains, on top, the information about the type of scalar product – its symmetry and the anti-involution associated with it. For quaternionic products, $\bar{\mathbb{H}}$ denotes that the anti-involution is quaternionic-Hermitian conjugation[9]. Below the type of product, the dimension of the vector space is given and, where applicable, the Witt index of the product. Higher-dimensional cases can be found using Bott periodicity. On the bottom

[8] See [8, Sec. 2.6] or [117, Ch. 17]

[9] In the mathematical literature, $\bar{\mathbb{H}}$-skew products are usually called $\hat{\mathbb{H}}$-symmetric, with $\hat{}$ quaternion reversion, $k \mapsto -k$.

of the field is given the algebra element (if any) that relates Clifford transposition and the adjoint, cf. (5.2). The table shows right away that some scalar products are unsuitable for physics: In physics, the starting point is always a irreducible representation, and one cannot use the scalar products marked by dotted-line boxes, where the adjoint mixes the two irreducible components. Of the remaining scalar products, it is advantageous to use those where the adjoint and Clifford transposition are related in the Euclidean case ($p = 0$ or $q = 0$) by unity, and in the space-time case ($p = 1$ or $q = 1$) by either e_1 or e_n. Clifford transposition can be chosen, as the tabular representations prove, to correspond to Hermitian conjugation, and with this, the aforementioned scalar products are just those in which the adjoint is, in the Euclidean case, Hermitian conjugation, and in the space-time case, Dirac conjugation $\bar{\Psi} := \Psi^\dagger\gamma^0$. This is a non-trivial property of the quaternions – in the case of Clifford algebras isomorphic to complex matrix algebras, some of the scalar products cannot be expressed this way, as their associated (anti-)involution is not complex conjugation, but the identity map.

Finally, one can write down the matrix equivalent of the abstract algebraic Fierz identity. Denoting the general representation matrix by A and the matrix corresponding to the canonical basis element $e_{i_1\ldots i_k}$ by $\gamma_{i_1\ldots i_k}$, and concentrating on irreducible representations (with suitable corrections for semi-simplicity),

$$A = \frac{1}{D}\sum_{k=0}^{N}\left[\frac{1}{k!}\mathrm{Re}\;\mathrm{Tr}\left(\gamma^{i_k\cdots i_1}A\right)\gamma_{i_1\ldots i_k}\right], \tag{5.13}$$

where $N = n$ for the simple and $N = [n/2]$ for the semi-simple case, and D is the dimension of the matrix algebra. The fact that the scalar projection is not mapped to the trace, but contains a reality condition, as well, is crucial – it means that, in contradistinction to a formulation of quaternionic Clifford algebras with complex matrices and a complex trace [113], the Fierz identity can be rewritten in the usual biquadratic form used in physics, thanks to the cyclicity property of quaternions, $p, q, r \in \mathbb{H} : \mathrm{Re}(pqr) = \mathrm{Re}(qrp)$: let ψ_A denote column spinor components and $\overline{\psi}_A$ line spinor components (in physics, $\overline{\psi} = \psi^\dagger\gamma^0$, with † Hermitian conjugation; this can be carried over to the case under study by using quaternionic conjugation). The type of quaternionic matrix of interest here has the bispinorial form $\psi_1\overline{\psi}_2$; applying (5.13), $\mathrm{Re}\,\mathrm{Tr}(\gamma^{i_k\cdots i_1}\psi_1\overline{\psi}_2)$ can be rewritten as $\mathrm{Re}(\overline{\psi}_2\gamma^{i_k\cdots i_1}\psi_1)$ despite the non-commutative nature of the quaternions.

The language gap between quaternionic Clifford algebras used in mathematically oriented text-books on spinors [89, 117, 14, 8] and the pseudoreal or symplectic spinor formulation routinely used by physicists [133, 19, 139] was first bridged by a side-result in a seminal article by Kugo and Townsend [87], where the authors show the mapping between symplectic spinors in space-times with signature 4 (explicitly: six-dimensional space-time) and a quaternionic formulation. The tabular representations derived in the preceding section, together with the properties of the "good" quaternionic Clifford algebras, show how the mapping can be extended to those algebras. I will take the conservative ("minimally quaternionic") approach of projecting the spinor product to the real numbers (which (5.13) suggests as a necessary ingredient for building higher-than-quadratic terms in the fermions related by Fierz identities) and keeping all non-spinorial fields real. One direct consequence of the projected scalar product is the presence of an additional symmetry: Right multiplication of two spinors $\psi_{1,2}$ with a unit quaternion q leaves the product invariant, Re $(\overline{\psi_1 q}\psi_2 q) =$ Re $(\overline{\psi_1}\psi_2)$, corresponding to an invariance under the multiplicative group of unit quaternions, $U(1,\mathbb{H}) \simeq SU(2,\mathbb{C}) \simeq USp(2,\mathbb{C})$ and hence to a simple symplectic symmetry. Starting point for the relation between symplectic spinors and quaternions is the well-known isomorphism between the quaternions and a subalgebra of $\mathcal{M}(2,\mathbb{C})$,

$$\Psi = \Psi_1 + i\Psi_2 + j\Psi_3 + k\Psi_4 \mapsto \begin{pmatrix} \Psi_1 - i\Psi_2 & \Psi_3 + i\Psi_4 \\ -\Psi_3 + i\Psi_4 & \Psi_1 - i\Psi_2 \end{pmatrix} =: M_\Psi, \qquad (5.14)$$

where the subalgebra in question can be characterized as follows: For $J := -\sigma_2$, (5.14) is an isomorphism $\mathbb{H} \simeq \{M \in \mathcal{M}(2,\mathbb{C}) | M^* = JMJ^{-1}\}$. As a direct consequence, quaternionic conjugation on a quaternion Ψ corresponds to Hermitian conjugation on the corresponding matrix M_Ψ, $M_{\bar{\Psi}} = (M_\Psi)^\dagger$. Furthermore, the real part of a quaternion is half the trace of the corresponding matrix, $\mathrm{Tr}\, M_\Psi = 2 \cdot \mathrm{Re}(\Psi)$. In search for a vector space for quaternions to act on by left multiplication, a natural candidate is the vector space of the quaternions themselves. However, the map (5.14) carries this over not to complex-valued vectors (the most natural choice of a linear space for $\mathcal{M}(2,\mathbb{C})$ to act on), but to complex matrices. One can view these matrices as a set of two complex-valued vectors Ψ^i, with one extra index $i = 1,2$, indicating which matrix column to which the vector corresponds. Again, not all pairs of complex vectors are allowed, as the vectors have inherited the condition $M^* = JMJ^{-1}$ from their matrix precursors. Introducing the convention that an index i is automatically lowered by complex conjugation, and noting that $(J)_{ab} = -\varepsilon_{ab}$ (where ε_{ab} is totally antisymmetric and $\varepsilon_{12} = +1$), the condition on the vectors becomes $(\Psi^*)_i = -\varepsilon_{ij} J\Psi^j$. This is the symplectic reality condition.

From the isomorphism (5.14), further relations can be derived. First of all, the spinor product with the symplectic indices contracted. It can be translated to the trace of the product of the corresponding matrices, and thus to twice the real part of the product of the corresponding quaternions,

$$(\bar{\psi}_i \phi^i) = Tr(M^\dagger{}_\psi \cdot M_\phi) \simeq 2\,\mathrm{Re}(\bar{\psi}\phi). \qquad (5.15)$$

Secondly, the product $\psi^i(\bar{\phi}_i \chi^j)$ of three symplectic spinors linked by cross-wise contraction of spinor and symplectic indices can be translated into a product of the corresponding matrices and hence the corresponding quaternions,

$$\psi^i(\bar{\phi}_i \chi^j) \simeq M_\psi \cdot M^\dagger{}_\phi \cdot M_\chi \simeq \psi\bar{\phi}\chi. \qquad (5.16)$$

The tabular representations show how this isomorphism can be carried over to all the quaternionic cases considered there: All that is added to an initial mapping of quaternions are extra tensor factors of real matrices, enlarging the dimension of the quaternionic matrices, but leaving the reality conditions unchanged. The table of scalar products tells us that the spin-invariant products needed can be constructed with the help of Clifford transposition, which, on the representation matrices, is just the Hermitian conjugation we have used in the above isomorphism.

So far, there has been no mention of the fact that spinors, in a physically sensible field theory, will have anti-commuting (Grassmann) components. This introduces an additional complex structure, namely that on the Grassmann variables (where the convention is chosen that defines Grassmann complex conjugation as an anti-involution). Let us take the spinor adjoint to act simultaneously as quaternionic conjugation on quaternions, as Hermitian conjugation on the group generators, and as Grassmann conjugation on the Grassmann components. However, the projection of the spinor product to its real part operates on the quaternionic structure only – a subtlety necessary to match the usual treatment of complex Grassmann spinors.

With these preparations, one can write down minimally quaternionic versions of $\mathcal{N} = 2$ models, or rewrite models with $Sp(2)$-symplectic spinors. For example, with these conventions, one can reproduce the $d = 5, \mathcal{N} = 2$ scalar product with symmetry (4.1), following directly from the real Clifford algebra $\mathcal{C}(1, 4)$. While one might think that this minimal quaternionic reformulation is merely a rewriting of physical concepts already known, its merit merely in making the quaternionic structures explicit, and thus showing the background of certain instances where such structures occur in physics (as in the example of R-symmetry described at the start of this analysis), this is only true up to a point: For multilinear spinor expressions, the mapping (5.16) is possible only because of the transposition inherent in the spinor conjugation. While this is sufficient to ensure that the usual terms occurring in a Lagrangian, or in its supersymmetry variations, can be mapped, there is no such map for products of spinor components such as can be expected in superspace formulations. Indeed, in one application of the isomorphism given by Townsend and Kugo, a superspace formulation is developed that explicitly cannot be expressed in terms of symplectic spinor components [17]. This raises the possibility that superspace formulations arising from quaternionic formulations such as the "minimally quaternionic" one presented here might have physical applications beyond bridging the "language gap" to mathematics.

Chapter 6

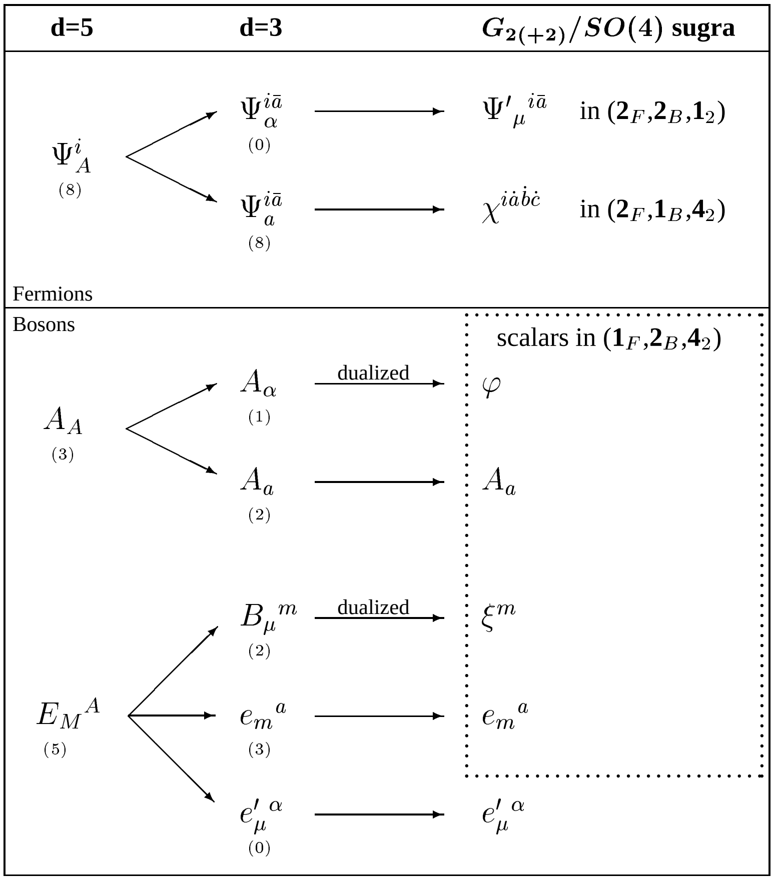
Summary and Outlook

In chapter 2, I have constructed the $G_{2(+2)}/SO(4)$ supergravity in three dimensions up to and including quadratic fermionic terms in the Lagrangian. The construction involved representations of $G_{2(+2)}$ decomposed relative to its maximal compact subgroup $SO(4)$ that were developed in appendix B. An analysis of the supersymmetry variations showed up an unusual symmetry structure that distinguishes this model from, for instance, its "big brother" $E_{8(+8)}/SO(16)$: In that model, the local $SO(16)$ used in the coset construction could be found directly from an analysis of the supercharges. Here, the analysis of the supercharges involves two $SO(3)$ groups; in a Fock construction of the multiplet, they are one local $SO(3)_B$ acting only on the bosons and one global $SO(3)_F$ acting only on the fermions. The coset construction, however, involves an additional local $SO(3)_2$ that does not act on the supercharges. From the analysis of the supersymmetry variations follows an action of those three groups on the gravitino Ψ (and the supersymmetry parameter ϵ), the spin-1/2 fermions χ and the scalars φ that ensures the necessary equality of bosonic and fermionic degrees of freedom as well as the necessary supersymmetry transformation behaviour by assigning these fields to the representations

	$SO(3)_F$	$SO(3)_B$	$SO(3)_2$
Ψ/ϵ	**2**	**2**	**1**
φ	**1**	**2**	**4**
χ	**2**	**1**	**4**

(recall that, in three dimensions, a spin-1/2 fermion and a scalar each have one physical degree of freedom; a graviton and gravitino have none).

Both the results of this chapter and the general expressions concerning dimensional splits of supergravity theories derived in chapter 3 paved the way for chapter 4, which dealt with the dimensional reduction of $d = 5, \mathcal{N} = 2$ supergravity to $d = 3$. By comparison with the three-dimensional model previously constructed, the hidden symmetry $G_{2(+2)}/SO(4)$ of the dimensionally reduced theory was uncovered. From the analysis of the supersymmetry variations and of the quadratic terms of the Lagrangian, the mapping of bosonic as well as fermionic fields between the two models followed. The results are summarized in the following table:



The leftmost column shows the fields of the $d = 5, \mathcal{N} = 2$ supergravity – the gravitino, the Maxwell field and the vielbein. The middle column shows the component fields resulting from the dimensional reduction. Below the fields in parentheses, the physical degrees of freedom they represent are shown. The rightmost column shows the redefinitions for the $G_{2(+2)}/SO(4)$ model. The dotted box contains the scalars that parametrize the coset. The pertinent representations with respect to the $SO(3)_F$, $SO(3)_B$, and $SO(3)_2$ are given in the form $(\mathbf{x}_F, \mathbf{y}_B, \mathbf{z}_2)$.

In contrast to the $E_{8(+8)}/SO(16)$ case, part of the enlargement of the local symmetry can be carried out explicitly: For the compound connections Q_μ, the anholonomy coefficients and F-term can be combined explicitly into an $SO(3)$-covariant form, using the three-dimensional gamma matrices $\hat{\Gamma}$ defined from the gamma matrices of the internal space. On the other hand, there is the unusual feature that the $SO(3)_2$ and the $SO(3)_B$ that make up the local symmetry occur, upon dimensional reduction, only in the form of their diagonal subgroup. Part of the enlargement process is the passing from this diagonal subgroup to the full $SO(3)_2 \times SO(3)_B$ needed in the construction of the coset model.

As a mathematical afterthought, I have taken a close look at quaternionic spinors, from the abstract algebra to tabular representations and the use of "minimal" explicitly quaternionic formulations in field theory.

To close, I give three examples for directions in which the results presented in this thesis could be extended.

Catching up with E_8: Lifting of symmetries

The first application of this thesis' results follows directly from one of the motivations I had given in the introduction. With the fermionic redefinitions that have been identified, it should be possible to lift the local hidden symmetry to the tangent space, just as in the eleven-dimensional theory [40, 104, 48]. If the generalized vielbeins introduced in the process show the same hints of an "exceptional geometry" as in eleven dimensions [85, 83], this would constitute a valuable toy model. Due to the much lower group dimension of G_2 (14 in comparison with the 248 of E_8), the calculations involving the matrices that are the generalized vielbeins should prove substantially simpler than for E_8. In this context, an investigation of the embedding of the G_2-model in the E_8 model, exposing the direct ties between the models, could prove useful.

Gauged supergravity

While the exceptional geometry would seek to highlight the similarities, it might prove instructive to exploit the differences to the eleven-dimensional case. Namely, in the five-dimensional $\mathcal{N} = 2$ theory, the R-symmetry, or a subgroup thereof, can be gauged [37, 64], and it would be interesting to see what happens to the hidden symmetries in these cases – which aspects of the G_2 symmetry can be preserved, and which are broken in the three-dimensional reduction? Added incentive is provided by the possibility of making contact with recent, more general, studies of the possible gaugings in three dimensions [110].

Compactification on AdS

So far, all compactification has taken place in the simplest possible way, namely on a hypertorus. There is another possibility that has great appeal; the compactification on special solutions $AdS_3 \times S^2$ which, for a certain relation between the cosmological constant of the AdS and the radius of the S^2, is a maximally supersymmetric vacuum of the theory – the analogy of the $AdS_7 \times S^4$ compactification of eleven-dimensional supergravity, and thus another possibility to make contact between five and eleven dimensions. The Kaluza-Klein spectrum of the reduction has already been derived in [56]. A further study of this compactification is of special interest because of its connections with the AdS/CFT-correspondence – either through the compactification of M-theory, wrapping the M5-branes on a four-cycle of one of a number of possible six-dimensional compactification spaces, obtaining a near-horizon geometry $AdS_3 \times S^2 \times M^6$ [2, Sect. 5.4], or, starting from a five-dimensional model, with a conjectured duality of the supergravity compactified on $AdS_3 \times S^2$ with a two-dimensional superconformal field theory living on the boundary of the AdS_3 [129].

Appendix A

Helpful auxiliary formulae

A.1 Coset models

The following are some relations that have been used in section 2.1. As in that section, assume that we are dealing with explicit matrix representations of G and H as well as of their Lie algebras. The Lie algebra adjoint map ad: Lie $G \to$ End(Lie G) is defined by ad $: X \mapsto \mathrm{ad}_X \equiv \mathrm{ad}(X) = [X, .]$. Furthermore, define the Lie group adjoint Ad $: G \to \mathrm{Aut}(G)$, by setting $\mathrm{Ad}(g) : G \to G$ to be the map that acts on a group element g_0 by $g_0 \mapsto g\, g_0\, g^{-1}$. The group adjoint induces a corresponding map on the Lie algebra, $Y \mapsto gYg^{-1}$ (for matrix representations, this is evidently well-defined). The two maps are related by[1]

$$\exp(X)Y\exp(-X) = \exp[\mathrm{ad}(X)]Y, , \tag{A.1}$$

reflecting the usual procedure of obtaining Lie group elements by the exponentiation of Lie algebra elements. With the help of this equality, it is possible to derive a useful relation between a small variation of a Lie algebra element X and the variation of the Lie group element $\exp(X)$ related to X via the exponential map, namely

$$e^{-X}\,\delta e^{X} = \frac{1-\exp[-\mathrm{ad}_X]}{\mathrm{ad}_X}\,\delta X. \tag{A.2}$$

Note that the right hand side is to be understood as defined by a power series,

$$\frac{1-\exp[-\mathrm{ad}_X]}{\mathrm{ad}_X} = \frac{1-\sum_{k=0}^{\infty}(-\mathrm{ad}_X)^k/k!}{\mathrm{ad}_X} = \sum_{k=1}^{\infty}\frac{(-\mathrm{ad}_X)^{k-1}}{k!}, \tag{A.3}$$

which is well-defined even if ad_X is not invertible. Eq. (A.2) can be proved in steps, as

$$e^{-X}\,\delta e^{X} = \int_0^1 e^{-tX}\,\delta X\,e^{tX}\,\mathrm{d}t = \frac{1-\exp[-\mathrm{ad}_X]}{\mathrm{ad}_X}\,\delta X. \tag{A.4}$$

[1] One way to prove the equation is to replace X by tX, with t some scalar variable, and Taylor-expand both sides around $t=0$, cf. [67, §I.4].

To prove the first equation, expand the left hand side and the integral term into power series, perform the integration, and re-sum using the identity

$$\binom{n}{m} = \sum_{k=0}^{m} \binom{n+1}{n-m+k+1} (-)^k. \tag{A.5}$$

For the second equation, use (A.1) to rewrite the integrand. Integration is now possible and gives the right hand side.

These relations can be used to calculate the explicit $\Sigma(\bar{g})$, the compensating H-transformation that is used to bring back an element of the coset into its properly gauged form after it has been acted upon from the left by a group element $\bar{g}$. Let $\bar{g} = e^{\alpha+\beta}$ with α shorthand for the non-compact part of the transformation, $\alpha := \alpha \cdot Y$, and β for its compact part $\beta \cdot X$. In the same vein, introduce the abbreviation $\varphi := \varphi \cdot Y$. Start with the transformed group element $g \mapsto \bar{g} g e^{\Sigma(h)}$. For the infinitesimal version of this transformation, it can be read off that

$$\delta g = \delta e^{\varphi} = (\alpha + \beta) e^{\varphi} + e^{\varphi} \Sigma. \tag{A.6}$$

Multiplication with g^{-1} gives us

$$e^{-\varphi}(\delta e^{\varphi}) = e^{-\varphi}(\alpha + \beta) e^{\varphi} + \Sigma.$$

Using eqq. (A.1) and (A.2), respectively, we can rewrite this as

$$\frac{1 - \exp[-\mathrm{ad}_{\varphi}]}{\mathrm{ad}_{\varphi}} \, \delta\varphi = e^{-\mathrm{ad}_{\varphi}} (\alpha + \beta) + \Sigma,$$

where ad_{φ} is the adjoint map acting on any algebra element X as $\mathrm{ad}_{\varphi}(X) = [\varphi, X]$. Viewed as a power series, the operator acting on $\delta\varphi$ on the left hand side is invertible. Acting with its inversion, it is possible to obtain a pure $\delta\varphi$ on the left hand side. Per definition of Σ, $\delta\varphi$ must be a linear combination of non-compact generators. Sorting the new right hand side into compact and non-compact parts by making use of the Lie algebra relations (2.1), one can obtain the two equations

$$\begin{aligned} \Sigma &= \tanh(\mathrm{ad}_{\varphi}/2)\alpha - \beta, \\ \delta\varphi &= \frac{\mathrm{ad}_{\varphi}}{\tanh(\mathrm{ad}_{\varphi})} \alpha - \mathrm{ad}_{\varphi} \beta \end{aligned} \tag{A.7}$$

given in the main text as (2.3) and (2.4), respectively.
Next, for the kinetic term. By (A.2), it holds that

$$e^{-\varphi} \partial_{\mu} e^{\varphi} = \frac{1 - \exp[-\mathrm{ad}_{\varphi}]}{\mathrm{ad}_{\varphi}} \, \partial_{\mu}\varphi. \tag{A.8}$$

Whether a term in the power series that is the right hand side of this equation is proportional to a linear combination of compact or of non-compact generators depends only on the power of ad $_\varphi$ mod 2. Separating even and odd powers,

$$\begin{aligned} P_\mu &= \frac{1}{2}(e^{-\varphi}\partial_\mu e^{\varphi} - e^{\varphi}\partial_\mu e^{-\varphi}), \\ Q_\mu &= \frac{1}{2}(e^{-\varphi}\partial_\mu e^{\varphi} + e^{\varphi}\partial_\mu e^{-\varphi}), \end{aligned} \tag{A.9}$$

or explicitly decomposing the power series on the right hand side of (A.8), the expressions given in the main text in eq. (2.5).

In the calculation of the supersymmetry variations of P_μ and Q_μ, one can start with eqq. (A.9) for those two objects, and then vary them using equation (A.2) and inserting (2.10), with the result being the expression (2.11) given in the main text.

A.2 Differential geometry

In this section, I give some basic formulae and expressions from differential geometry, used in the main text. Descriptions of supergravity, with their need to describe bosonic and fermionic degrees of freedom in curved space-time, usually employ what is called the *vielbein formalism* or *orthonormal frame* formalism[2]. It amounts to a choice of orthonormal vectors in the tangent spaces of the space-time manifold, $E_A = E_A{}^M\partial_M$, with the indices $A, B, \ldots$ labeling the vectors and $M, N, \ldots$ the indices associated with the coordinate basis. With the help of the coefficient fields $E_A{}^M$ and their matrix inverses, the vielbeins $E_M{}^A$, it is possible to express the metric with the help of the manifold as $g_{MN} = e_M{}^A e_N{}^B \eta_{AB}$, with η_{AB} a flat Minkowski metric incorporating the orthonormality of this basis. The vielbeins allow one to “change curved spacetime indices to flat indices”, i.e. express vectors or forms in terms of components relative to the orthogonal basis. The choice of basis is not unique; on the contrary, any Lorentz transformation in a tangent space, acting on flat indices, leaves η_{AB} invariant, and the most general possibility for changing the basis fields involve a local Lorentz transformation acting differently in the tangent spaces associated with different spacetime points. For these local Lorentz transformations, one can introduce a connection ω_{MAB} and its associated covariant derivative, acting on flat indices as

$$(D_M v)^A = \partial_M v^A + \omega_M{}^A{}_B v^B \quad \text{and} \quad (D_M\theta)_A = \partial_M\theta_A - \omega_M{}^B{}_A\theta^B. \tag{A.10}$$

What is especially attractive is the possibility of linking this Lorentz-covariant derivative with the usual (metric) covariant derivative defined on the curved space-time. This is achieved by demanding that the two derivatives be linked by the transitions from curved to flat or from flat to curved indices: The metric covariant derivative, acting

[2]For an introduction, see for example [136], [61, Ch. 12] or, in a more general geometrical context, [102].

on a vector specified by its curved components and the result transformed to flat components, should be the same as first transforming the original vector components into flat components, and then acting with the Lorentz covariant derivative. This is called the *vielbein postulate*, and it allows the reconstruction of the connection in question, which is called the *spin connection*, in terms of the vielbein as

$$\omega_{MAB} = E_{[A}{}^N \nabla_{|M} E_{N|B]}. \tag{A.11}$$

With the components of the torsion T, the tensor defined in terms of the metric connection and the Lie bracket $[v,w]^N = v^M(\partial_M w^N) - w^M(\partial_M v^N)$ of vectors by its action to map two vectors v, w to another vector $T(v,w) = \nabla_v w - \nabla_w v - [v,w]$, we can define the components of the contorsion

$$K^M{}_{NR} := \frac{1}{2}(T^M{}_{NR} + T_R{}^M{}_N + T_N{}^M{}_R). \tag{A.12}$$

Rewriting those in terms of flat components, one can solve the vielbein constraint (A.11) to obtain

$$\omega_{MAB} = \frac{1}{2} E_M{}^C \left[\Omega_{ABC} - \Omega_{BCA} - \Omega_{CAB}\right] \; + \; K_{AMB}, \tag{A.13}$$

used in section 3.1.

Modulo global issues that are far beyond the scope of this appendix, the vielbein formalism allows the definition of spinor fields in curved space-time. The Lorentz group acting on flat indices has well-known spinor representations, and one can carry over the ample knowledge one has about spinors in flat space-time and, for instance, define gamma matrices Γ^A satisfying a Clifford algebra relation. For many interesting cases, the spinors thus defined in the tangent spaces can be knit together to spinor fields living in space-time, and all the spinor fields occuring in this thesis are defined in this way. The Lorentz covariant derivative has a natural extension to the spinors – after all, one can formulate the Lorentz generators in terms of products of gamma matrices –, namely

$$D_M \chi = \partial_M \chi + \frac{1}{4} \omega_{MAB} \Gamma^{AB} \chi. \tag{A.14}$$

With this definition, it is possible to write down covariant Lagrangians and equations of motion for the spinor fields, just as we have done many a time in the main text.

Finally, it is possible to rewrite the objects defining the space-time geometry – notably curvature and torsion – in terms of vielbein and spin connection. There are a number of different conventions for defining the Riemann curvature and contracting it to the Ricci tensor and curvature scalar. Define the curvature as

$$R^A{}_{BMN} \;=\; \partial_M(\omega_N{}^A{}_B) - \partial_N(\omega_M{}^A{}_B) + \omega_N{}^C{}_B \omega_M{}^A{}_C - \omega_M{}^C{}_B \omega_N{}^A{}_C,$$

and the components of the Ricci tensor and the curvature scalar, respectively,

$$Ric_{BN} = E_A{}^M R^A{}_{BMN} \quad \text{and} \quad \mathcal{R} = E^{BN}\, Ric_{BN}. \tag{A.15}$$

With these definitions, it can be shown that, except for torsion contributions,

$$\begin{aligned} E\mathcal{R} &= \frac{1}{4}E\left[-\Omega_{ABC}\Omega^{ABC} + 2\,\Omega_{ABC}\Omega^{CAB} + 4\,\Omega_{AB}{}^{B}\Omega^{A}{}_{C}{}^{C}\right] \\ &\quad +\partial_M(2\,E\,E^{AM}\Omega_{AB}{}^{B}). \end{aligned} \tag{A.16}$$

A.3 d=5, N=2 supergravity

In section 4.1, the gamma matrices of the $\mathcal{C}_c(1,4)$ are defined in terms of the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.17}$$

as

$$\begin{aligned} \Gamma^0 &= -\epsilon_3\sigma_2 \otimes \sigma_2 \\ \Gamma^1 &= i\sigma_1 \otimes \sigma_2 \\ \Gamma^2 &= i\sigma_3 \otimes \sigma_2 \\ \Gamma^3 &= i\mathbb{1} \otimes \sigma_1 \\ \Gamma^4 &= i\mathbb{1} \otimes \sigma_3. \end{aligned} \tag{A.18}$$

For the derivation of the Fierz identity (4.5), the duality relations

$$\begin{aligned} \Gamma_{ABCDE} &= \epsilon_3\varepsilon_{ABCDE} \\ \Gamma_{ABCD} &= \epsilon_3\varepsilon_{ABCDE}\,\Gamma^{E} \\ \Gamma_{ABC} &= -\frac{1}{2}\,\epsilon_3\varepsilon_{ABCDE}\,\Gamma^{DE} \end{aligned} \tag{A.19}$$

between the five-dimensional gamma matrices were used. For the derivation of Fierz identities, it is easiest to hide the symplectic indices away by defining $\varphi := \Psi^1$ and $\chi := \Psi^2$ and writing any non-vanishing contracted spinor product as

$$(\overline{\Psi}_i\Gamma_{A_1\ldots A_m}\Psi^i) = 2(\overline{\varphi}\Gamma_{A_1\ldots A_m}\varphi) = 2(\overline{\chi}\Gamma_{A_1\ldots A_m}\chi),$$

using the symmetry condition (4.4). Whenever, using this identity, we succeed in rewriting the terms in the putative Fierz identity solely in terms of φ or solely in terms of χ, a proof will be much easier than it would be if one had to keep track of all the symplectic indices, as well. For instance, in this manner, the Fierz identity (4.5) is derived as follows: Rewriting the cases $i = 1, 2$ with the help of φ and χ leads to one and the same Fierz identity – once for φ, once for χ. In order to prove it, apply (4.5) first to $A := \Gamma_{[M}\varphi_N(\overline{\varphi}_P\varphi_{Q]})$, then to $B := \varphi_{[M}(\overline{\varphi}_N\Gamma_P\varphi_{Q]}$. The latter contains $C := \Gamma^A\varphi_{[M}(\overline{\varphi}_N\Gamma_{|A|P}\varphi_{Q]})$, which in turn can be transformed in a third application of (4.5). Substituting the fierzed C into the fierzed B and subtracting the fierzed A gives the desired identity (4.5).

Appendix B

Some useful representations for $\mathfrak{g}_2$

The algebraic objects playing the central role in this thesis are the exceptional Lie group G_2 and its Lie algebra $\mathfrak{g}_2$, first discovered by Killing at the end of the 19th century, during his ground-breaking work on the classification of simple Lie algebras [77, 78, 79, 80].

In this appendix, some useful representations of the Lie algebra $\mathfrak{g}_2$ are constructed in terms of tensor representations of $\mathfrak{so}(4,\mathbb{R}) \simeq \mathfrak{so}(3,\mathbb{R}) + \mathfrak{so}(3,\mathbb{R})$, providing the group theoretical input needed in chapter 2 for the construction of the corresponding coset model.

From the literature [94] or, equivalently, from the available group theory software[1] one can take the following basic facts: The adjoint representation of the Lie algebra g_2 is the representation **14** (here and in the following, I denote the representations involved by their dimension, written in boldface). Under $\mathfrak{so}(4,\mathbb{R})$, this representation decomposes as $\mathbf{14} = \mathbf{3} + \mathbf{3}' + \mathbf{8}$ or, showing the representations of the two $\mathfrak{so}(3,\mathbb{R})$ that make up $\mathfrak{so}(4,\mathbb{R})$, as $\mathbf{14} = (\mathbf{1},\mathbf{3}) + (\mathbf{3},\mathbf{1}) + (\mathbf{4},\mathbf{2})$.

With this information, it is possible to build the **14** tensor representation. Let $\bar{a}, \bar{b}, \ldots$ be indices of the fundamental (i.e. spinor) representation **2** of the first $\mathfrak{so}(3,\mathbb{R})$, showing up in the decomposition as $(\mathbf{1},\mathbf{3})$, and $\dot{a}, \dot{b}, \ldots$ those of the second, the $(\mathbf{3},\mathbf{1})$. Irreducible $\mathfrak{so}(3,\mathbb{R})$ tensors are either symmetric in all their indices or can be obtained from symmetric tensors by shifting indices via contraction by the antisymmetric Levi-Civita-tensor, which converts symmetry properties into conditions of tracelessness. Define the tensors that are to make up the representation as follows: The antisymmetric invariant $\varepsilon^{\dot{a}\dot{b}}$ is defined by $\varepsilon^{12} = +1$, its counterpart $\varepsilon_{\dot{a}\dot{b}}$ by $\varepsilon_{\dot{a}\dot{b}}\varepsilon^{\dot{b}\dot{c}} = -\delta^{\dot{c}}_{\dot{a}}$. Corresponding definitions are made for the other $\mathfrak{so}(3,\mathbb{R})$. Objects in the $(\mathbf{1},\mathbf{3})$ part of the **14** shall have the index structure $M^{\dot{a}}{}_{\dot{b}}$ and are taken to be traceless; similarly, the objects in the $(\mathbf{3},\mathbf{1})$ are traceless tensors $N^{\bar{a}}{}_{\bar{b}}$. Objects transforming as $(\mathbf{2},\mathbf{4})$ are tensors $Y^{\bar{a}\dot{a}\dot{b}\dot{c}}$ totally symmetric in $\dot{a}, \dot{b}, \dot{c}$.

It proves useful to define objects with lowered indices as well; in particular, let us

[1] For example, A.M. Cohen, M.A.A. van Leeuwen and B. Lisser's software package *LiE*, available online at [`http://young.sp2mi.univ-poitiers.fr/~marc/LiE/`].

define lower-index objects in the $(\mathbf{2},\mathbf{4})$ by

$$Y_{\bar{a}\dot{a}\dot{b}\dot{c}} := \varepsilon_{\bar{a}\bar{b}}\varepsilon_{\dot{a}\dot{d}}\varepsilon_{\dot{b}\dot{e}}\varepsilon_{\dot{c}\dot{f}}Y^{\bar{b}\dot{d}\dot{e}\dot{f}}.$$

From the index structure and standard tensor product decomposition follows the structure of the possible commutators between these objects, up to proportionality constants. Some of those constants are fixed by the mandatory Jacobi identities, and as a result, the commutation relations take on the form

$$\begin{aligned}
[M,M']^{\dot{a}}{}_{\dot{b}} &= a\cdot(M^{\dot{a}}{}_{\dot{c}}M'^{\dot{c}}{}_{\dot{b}} - M'^{\dot{a}}{}_{\dot{c}}M^{\dot{c}}{}_{\dot{b}})\\
[M,Y]^{\bar{a}\dot{a}\dot{b}\dot{c}} &= 3a\cdot Y^{\bar{a}\dot{d}(\dot{a}\dot{b}}M^{\dot{c})}{}_{\dot{d}}\\
[Y,Y']^{\dot{a}}{}_{\dot{b}} &= c(Y'^{\bar{a}\dot{a}\dot{c}\dot{d}}Y_{\bar{a}\dot{b}\dot{c}\dot{d}} - Y^{\bar{a}\dot{a}\dot{c}\dot{d}}Y'_{\bar{a}\dot{b}\dot{c}\dot{d}})\\
[M,N] &= 0\\
[N,N']^{\bar{a}}{}_{\bar{b}} &= a'\cdot(N^{\bar{a}}{}_{\bar{c}}N'^{\bar{c}}{}_{\bar{b}} - N'^{\bar{a}}{}_{\bar{c}}N^{\bar{c}}{}_{\bar{b}})\\
[N,Y]^{\bar{a}\dot{a}\dot{b}\dot{c}} &= a'\cdot N^{\bar{a}}{}_{\bar{c}}Y^{\bar{c}\dot{a}\dot{b}\dot{c}}\\
[Y,Y']^{\bar{a}}{}_{\bar{b}} &= c'(Y'^{\bar{a}\dot{a}\dot{b}\dot{c}}Y_{\bar{b}\dot{a}\dot{b}\dot{c}} - Y^{\bar{a}\dot{a}\dot{b}\dot{c}}Y'_{\bar{b}\dot{a}\dot{b}\dot{c}}),
\end{aligned} \tag{B.1}$$

where the constants a, a', c and c' are related by $ac = a'c'$. Next, one can check how the objects M, N and Y act on the fundamental representation **7** of $\mathfrak{g}_2$. That representation decomposes into two $\mathfrak{so}(3,\mathbb{R}) + \mathfrak{so}(3,\mathbb{R})$-representations as $\mathbf{7} = (\mathbf{2},\mathbf{2}) + (\mathbf{3},\mathbf{1})$. The latter have already been identified with traceless objects $w_{\dot{a}}{}^{\dot{b}}$; the former is made up of objects $\Psi^{\bar{a}\dot{a}}$ (corresponding to a vector of the $\mathfrak{so}(4,\mathbb{R})$). Again, tensor product decomposition gives us the proper index structure, and demanding that, for any two Lie algebra elements, the corresponding infinitesimal transformations satisfy $[\delta_X,\delta_Y] = \delta_{[Y,X]}$ (which corresponds to defining variations of the algebra elements themselves by $\delta_X Y = [Y,X]$), as well as comparison with (B.1) shows that the group elements M, N, Y act on an element $(\Psi^{\bar{a}\dot{a}}, w_{\dot{a}}{}^{\dot{b}})$ as

$$\begin{aligned}
\delta_M(\Psi^{\bar{a}\dot{a}}, w^{\dot{a}}{}_{\dot{b}}) &= (-aM^{\dot{a}}{}_{\dot{b}}\Psi^{\bar{a}\dot{b}}, a[w^{\dot{a}}{}_{\dot{c}}M^{\dot{c}}{}_{\dot{b}} - M^{\dot{a}}{}_{\dot{c}}w^{\dot{c}}{}_{\dot{b}}]),\\
\delta_N(\Psi^{\bar{a}\dot{a}}, w^{\dot{a}}{}_{\dot{b}}) &= (-a'N^{\bar{a}}{}_{\bar{b}}\Psi^{\bar{b}\dot{a}}, 0),\\
\delta_Y(\Psi^{\bar{a}\dot{a}}, w^{\dot{a}}{}_{\dot{b}}) &= (\bar{b}\,Y^{\bar{a}\dot{a}\dot{b}\dot{c}}w^{\dot{d}}{}_{\dot{c}}\varepsilon_{\dot{b}\dot{d}},\ \bar{d}\,\varepsilon^{\dot{a}\dot{c}}Y_{\bar{a}\dot{b}\dot{c}\dot{d}}\Psi^{\bar{a}\dot{d}}),
\end{aligned} \tag{B.2}$$

where a, a' are the same as in (B.1), while the new constants $\bar{b}$ and $\bar{d}$ must satisfy $\bar{b}\bar{d} = 2ac$.

So far, we have only talked about tensors in general. In order to specify the Killing form, one needs to define a set of generators. The most natural choice is that of generators with projective properties, as follows. Define

$$
\begin{aligned}
E^{\dot a}{}_{\dot b}|^{\dot c}{}_{\dot d} &= \delta^{\dot a}_{\dot d}\delta^{\dot c}_{\dot b} - \frac{1}{2}\delta^{\dot a}_{\dot b}\delta^{\dot c}_{\dot d} \\
E^{\bar a}{}_{\bar b}|^{\bar c}{}_{\bar d} &= \delta^{\bar a}_{\bar d}\delta^{\bar c}_{\bar b} - \frac{1}{2}\delta^{\bar a}_{\bar b}\delta^{\bar c}_{\bar d} \\
E_{\bar a\dot a\dot b\dot c}|^{\bar b\dot d\dot e\dot f} &= \delta^{\bar b}_{\bar a}\,\delta^{\dot d}_{(\dot a}\delta^{\dot e}_{\dot b}\delta^{\dot f}_{\dot c)},
\end{aligned}
\tag{B.3}
$$

where the first set of indices serves to label the generators, while the rightmost set identifies them as tensor representatives belonging to the $(\mathbf{3},\mathbf{1})$, $(\mathbf{1},\mathbf{3})$ and $(\mathbf{4},\mathbf{2})$ representations, respectively. By construction, the two index sets are interchangeable.

Note that the numbering of generators is not unique – for example, tracelessness dictates that $E^1{}_1| = -E^2{}_2|$. Thus, (B.3) gives an overcomplete set of generators, and one needs to keep in mind that "summing over (linearly independent) generators" is, in general, not the same as summing over all index combinations.

All the generators have projective properties: For $M^{\dot a}{}_{\dot b}$ traceless, contraction with a generator gives $M^{\dot a}{}_{\dot b}\;E^{\dot b}{}_{\dot a}|^{\dot c}{}_{\dot d} = M^{\dot c}{}_{\dot d}$; a corresponding relation holds for $E^{\bar a}{}_{\bar b}|^{\bar c}{}_{\bar d}$ and any traceless $N^{\bar a}{}_{\bar b}$; finally, for any object $Y^{\bar a\dot a\dot b\dot c}$ symmetric in $\dot a, \dot b, \dot c$, the relation $Y^{\bar a\dot a\dot b\dot c}E_{\bar a\dot a\dot b\dot c}|^{\bar b\dot d\dot e\dot f} = Y^{\bar b\dot d\dot e\dot f}$ holds.

These generators can be used to define a general algebra element as

$$
M + N + Y := M^{\dot a}{}_{\dot b}E^{\dot b}{}_{\dot a}| + N^{\bar a}{}_{\bar b}E^{\bar a}{}_{\bar b}| + Y^{\bar a\dot a\dot b\dot c}E_{\bar a\dot a\dot b\dot c}|,
$$

with the $M^{\dot a}{}_{\dot b}$, $N^{\bar a}{}_{\bar b}$ and $Y^{\bar a\dot a\dot b\dot c}$ coefficients that, without any loss of generality, can be taken to share the index properties of the corresponding algebra elements (i.e. $M^{\dot a}{}_{\dot b}$ and $N^{\bar a}{}_{\bar b}$ are traceless, $Y^{\bar a\dot a\dot b\dot c}$ is symmetric in $\dot a, \dot b, \dot c$). The ambiguity in notation, with the new coefficients $M^{\dot a}{}_{\dot b}$ denoted in exactly the same way as the algebra elements that were introduced previously, is no accident. Indeed, it follows from the projective property of the generators that the coefficients reproduce exactly the commutation relations (B.1): For example, for two algebra elements $M = M^{\dot a}{}_{\dot b}\;E^{\dot b}{}_{\dot a}|$ and $M' = M^{\dot a}{}_{\dot b}\;E^{\dot b}{}_{\dot a}|$, the coefficients defining the commutator $[M, M'] = [M, M']^{\dot a}{}_{\dot b}\;E^{\dot b}{}_{\dot a}|$ are just given by $[M, M']^{\dot a}{}_{\dot b} = a\cdot(M^{\dot a}{}_{\dot c}M'^{\dot c}{}_{\dot b} - M'^{\dot a}{}_{\dot c}M^{\dot c}{}_{\dot b})$.

It is straightforward to derive the structure constants associated with the generators (B.3), although one needs to be careful to take the inequivalence between summation over linearly independent generators and summing over indices into account.

With the help of the structure constants, one can calculate the coefficients of the Killing form in the usual way: For a set $\{T_a\}$ of linearly independent generators with $[T_a, T_b] = f_{ab}{}^cT_c$, and for κ the Killing form, the relation $\kappa(T_a, T_b) = f_{ac}{}^d f_{bd}{}^c$ holds.[2]

As a result, the action of the Killing form on two general algebra elements $A =$

[2]Proof can be found in textbooks dealing with the theory of Lie algebras, e.g. [55, Section 6.3].

$M + N + Y$ and $A' = M' + N' + Y'$ is

$$\begin{aligned}\kappa(A, A') &= 24a^2 M^{\dot a}{}_{\dot b} M'^{\dot b}{}_{\dot a} + 8(a')^2 N^{\bar b}{}_{\bar a} N'^{\bar a}{}_{\bar b} - 16ac\, Y_{\bar a \dot a \dot b \dot c} Y'^{\bar a \dot a \dot b \dot c} \\ &= 24a^2 \mathrm{Tr}\,(MM') + 8(a')^2 \mathrm{Tr}\,(NN') - 16ac\, Y_{\bar a \dot a \dot b \dot c} Y'^{\bar a \dot a \dot b \dot c}. \end{aligned} \quad \text{(B.4)}$$

While I have blithely talked of $\mathfrak{so}(3, \mathbb{R}) + \mathfrak{so}(3, \mathbb{R})$, I have so far not taken any steps to define what real form of $\mathfrak{g}_2$ it is that we are dealing with in the first place, and to ensure that the decomposition is indeed with respect to a maximal $\mathfrak{so}(3, \mathbb{R}) + \mathfrak{so}(3, \mathbb{R})$, and not some other real form. To remedy this situation, reality conditions on the objects M, N and Y are needed.

The simplest such condition, namely to define all coefficients $M^{\dot a}{}_{\dot b}$, $N^{\bar a}{}_{\bar b}$ and $Y^{\bar a \dot a \dot b \dot c}$ as well as the constants like a, c to be real, does not lead to the desired result: Inserting a set of linearly independent generators into (B.4) shows that, in this way, one obtains a representation of $\mathfrak{g}_2$ built from tensor representations of a maximal subalgebra $\mathfrak{sl}(2, \mathbb{R}) + \mathfrak{sl}(2, \mathbb{R})$.

Hence, more general reality conditions are needed. It turns out that the most general (linear) reality condition on the coefficients that is compatible with the commutation relations (B.1) is

$$\begin{aligned}(M^{\dot a}{}_{\dot b})^* &= (C)^{\dot a}{}_{\dot c} (\mathrm{Tr}\, C \cdot M^{\dot c}{}_{\dot b} - M^{\dot c}{}_{\dot d} (C)^{\dot d}{}_{\dot b}) \\ (N^{\bar a}{}_{\bar b})^* &= (C')^{\bar a}{}_{\bar c} (\mathrm{Tr}\, C' \cdot N^{\bar c}{}_{\bar b} - N^{\bar c}{}_{\bar d} (C')^{\bar d}{}_{\bar b}) \\ (Y^{\bar a \dot a \dot b \dot c})^* &= \mu (C')^{\bar a}{}_{\bar b} C_{\dot e}{}^{\dot a}\, C_{\dot f}{}^{\dot b}\, C_{\dot g}{}^{\dot c}\, Y^{\bar b \dot e \dot f \dot g}, \end{aligned} \quad \text{(B.5)}$$

provided that the following holds: the complex number μ satisfies

$$\mu^2 = \left(\frac{c'}{c'^*}\right) \left(\frac{a^*}{a}\right)^3. \quad \text{(B.6)}$$

The matrix C satisfies $(C^* C)_{\dot a}{}^{\dot b} = \epsilon_C \delta^{\dot b}_{\dot a}$, with ϵ_C equal to plus or minus one, and it is of the form

$$C = \sqrt{\frac{a}{a^*}} \begin{pmatrix} \sqrt{\epsilon_C - c_1 c_2}\, e^{\bar a \varphi} & c_1 \sqrt{-\epsilon_c} \\ c_2 \sqrt{-\epsilon_c} & \epsilon \sqrt{\epsilon_C - c_1 c_2}\, e^{-i\varphi} \end{pmatrix}. \quad \text{(B.7)}$$

In this expression, φ, c_1 and c_2 are real parameters. If $\epsilon_c = +1$, then c_1 and c_2 must satisfy $c_1 c_2 \leq 1$; for $\epsilon_c = -1$, $c_1 c_2 \leq -1$. The matrix C' satisfies the same conditions, save for the substitution of $a \to a'$, with possibly different parameters ϵ'_C, c'_1, c'_2 and φ'.

From the possibilities this opens up, now for a concrete choice. In addition, I introduce a new convention: Let the action of complex conjugation be linked with a change of index positions – define complex conjugate objects as having indices down where there counterparts have indices up, and conversely. Complex conjugation changes the index structure of purely real objects, as well: We have $(\varepsilon^{\dot{a}\dot{b}})^* = \varepsilon_{\dot{a}\dot{b}}$ and, for the generators, $(E^{\dot{a}}{}_{\dot{b}}|^{\dot{c}}{}_{\dot{d}})^* = E^{\dot{b}}{}_{\dot{a}}|^{\dot{d}}{}_{\dot{c}}$, $(E^{\bar{a}}{}_{\bar{b}}|^{\bar{c}}{}_{\bar{d}})^* = E^{\bar{b}}{}_{\bar{a}}|^{\bar{d}}{}_{\bar{c}}$ and $(E_{\bar{a}\dot{a}\dot{b}\dot{c}}|^{\bar{b}\dot{d}\dot{e}\dot{f}})^* = E_{\bar{b}\dot{d}\dot{e}\dot{f}}|^{\bar{a}\dot{a}\dot{b}\dot{c}}$. With this new convention, let us choose the reality conditions

$$\begin{aligned}
(M^{\dot{a}}{}_{\dot{b}})^* = (M^*)_{\dot{a}}{}^{\dot{b}} &= -\varepsilon_{\dot{a}\dot{c}}M^{\dot{c}}{}_{\dot{d}}\varepsilon^{\dot{d}\dot{b}} = -M^{\dot{b}}{}_{\dot{a}} \\
(N^{\bar{a}}{}_{\bar{b}})^* = (N^*)_{\bar{a}}{}^{\bar{b}} &= -\varepsilon_{\bar{a}\bar{c}}N^{\bar{c}}{}_{\bar{d}}\varepsilon^{\bar{d}\bar{b}} = -N^{\bar{b}}{}_{\bar{a}} \\
(Y^{\bar{a}\dot{a}\dot{b}\dot{c}})^* = (Y^*)_{\bar{a}\dot{a}\dot{b}\dot{c}} &= -\varepsilon_{\bar{a}\bar{b}}\varepsilon_{\dot{a}\dot{d}}\varepsilon_{\dot{b}\dot{e}}\varepsilon_{\dot{c}\dot{f}}Y^{\bar{b}\dot{d}\dot{e}\dot{f}} = -Y_{\bar{a}\dot{a}\dot{b}\dot{c}}.
\end{aligned} \tag{B.8}$$

They imply corresponding reality conditions on the constants a, c, a' and c', namely that all those constants are real. The action of the Killing form follows directly from (B.4), and we will only note that we can rewrite the block concerning the $(\mathbf{4}, \mathbf{2})$ as

$$\kappa(Y, Y') = -16ac\, Y_{\bar{a}\dot{a}\dot{b}\dot{c}}Y'^{\bar{a}\dot{a}\dot{b}\dot{c}} = 16ac\,(Y^*)_{\bar{a}\dot{a}\dot{b}\dot{c}}Y'^{\bar{a}\dot{a}\dot{b}\dot{c}}. \tag{B.9}$$

The last form involving complex conjugation is suitable for use with algebra-valued spinors, as spinor products traditionally involve complex conjugation of conjugated spinors.

Diagonalizing the Killing form, it turns out that the new reality condition indeed satisfies the requirements: In the corresponding real form of the algebra, the objects in the $(\mathbf{3}, \mathbf{1})$ and in the $(\mathbf{1}, \mathbf{3})$ generate one subalgebra $\mathfrak{so}(3, \mathbb{R})$, each, and, together with the objects in the $(\mathbf{4}, \mathbf{2})$, they generate the maximally non-compact form $\mathfrak{g}_{2(+2)}$ if $ac > 0$, and the compact form $\mathfrak{g}_{2(-14)}$ if $ac < 0$.

For consistency reasons, it is necessary to impose reality conditions on the objects from the vector space on which the representation **7** acts, as well. The $w^{\dot{a}}{}_{\dot{b}}$ in their $(\mathbf{3}, \mathbf{1})$, one can choose to obey the same reality condition as the $M^{\dot{a}}{}_{\dot{b}}$. For the $\Psi^{\bar{a}\dot{a}} \in (\mathbf{2}, \mathbf{2})$, one can demand

$$(\Psi^{\bar{a}\dot{a}})^* = (\Psi^*)_{\bar{a}\dot{a}} = -\varepsilon_{\bar{a}\bar{b}}\varepsilon_{\dot{a}\dot{b}}\Psi^{\bar{b}\dot{b}}. \tag{B.10}$$

It follows that, in order to achieve consistency, the constants $\bar{b}, \bar{d}$ in the transformation relations (B.2) must be real, as well

Now that the correct real form is found, there is an opportunity to make good use of the remaining freedom of rescaling algebra elements. With this, I choose $a = a' = c = c' = 1$, and $\bar{b} = \bar{d} = \sqrt{2}$ to make transformation rules and commutation relations especially simple.

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