ERROR–CORRECTING CODES AND PHASE TRANSITIONS

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Abstract. The theory of error-correcting codes is concerned with constructing codes that optimize simultaneously transmission rate and relative minimum distance. These conflicting requirements determine an asymptotic bound, which is a continuous curve in the space of parameters. The main goal of this paper is to relate the asymptotic bound to phase diagrams of quantum statistical mechanical systems. We first identify the code parameters with Hausdorff and von Neumann dimensions, by considering fractals consisting of infinite sequences of code words. We then construct operator algebras associated to individual codes. These are Toeplitz algebras with a time evolution for which the KMS state at critical temperature gives the Hausdorff measure on the corresponding fractal. We extend this construction to algebras associated to limit points of codes, with non-uniform multi-fractal measures, and to tensor products over varying parameters.

Contents.

0. Introduction: asymptotic bounds.

0.1. Notation. The following notation is used throughout the paper. *An alphabet* is a finite set A of cardinality $q \geq 2$, *a code* is a subset $C \subset A^n$, $n = n(C) \geq 1$. *Words of length* n are elements of A^n, they are generally denoted $(a_1, \dots, a_n)$, $a_i \in A$ and alike. Elements of C are *code words.*

The Hamming distance between two words (a_i), (b_i) is defined as

$$d((a_i),(b_i)) := \#\{i \in (1,\dots,n) \,|\, a_i \neq b_i\}.$$

The minimal distance $d = d(C)$ of the code C is

$$d(C) := \min \{d(a, b) \mid a, b \in C, a \neq b\}.$$

Finally, we put

$$k = k(C) := \log_q \#C, \quad [k] = [k(C)] := \text{integer part of } k(C),$$

so that

$$q^{[k]} \leq \#C = q^k < q^{[k]+1}. \tag{0.1}$$

The numbers n, k, d and q are called *parameters of* C, and a code C with such parameters is called an $[n, k, d]_q$–code. Notice that any bijective map between two alphabets produces a bijection between the associated sets of codes, preserving all code parameters.

Alphabet A and code C may be endowed with additional structures. The most popular case is: $A = \mathbf{F}_q$, the finite field with q elements, and C is a linear subspace of $\mathbf{F}_q^n$. Such codes are called *linear ones.*

Codes are used to transmit signals as sequences of code words. Encoding such a signal may become computationally more feasible, if the code is a structured set, such as a linear space. During the transmission, code words may be spoiled by a random noise, which randomly changes letters constituting such a word. The noise produces some word in A^n which might not belong to C. At the receiver end, the (conjecturally) sent word must be reconstructed, for example, as closest neighbor in C (in Hamming's metric) of the received word. This decoding operation again might become more computationally feasible, if A and C are endowed with an additional structure.

If k is small with respect to n, there are relatively few code words, and decoding becomes safer, but the price consists in the respective lengthening of the encoding signal. The number $R = R(C) := k/n,\ 0 < R \leq 1$, that measures the inverse of this lengthening, is called the (relative) *transmission rate.* If d is small, there might be too many code words close to the received word, and the decoding becomes less safe. The number $\delta := \delta(C) = d/n,\ 0 < \delta \leq 1$, is called *the relative minimal distance* of C.

The theory of error–correcting codes is concerned with studying and constructing codes C that satisfy three mutually conflicting requirements:

(i) Fast transmission rate $R(C)$*.*

(ii) Large relative minimal distance $\delta(C)$.

(iii) Computationally feasible algorithms of producing such codes, together with feasible algorithms of encoding and decoding.

As is usual in such cases, a sound theory must produce a picture of limitations, imposed by this conflict. The central notion here is that of *the asymptotic bound*, whose definition was given and existence proved in [Man]. The next subsection is devoted to this notion.

0.2. Code points and code domains. We first consider all $[n,k,d]_q$–codes C with fixed $q>1$ and varying n,k,d. To each such code we associate the point

$$P_C := (R(C),\delta(C)) = (k(C)/n(C), d(C)/n(C)) \in [0,1]^2.$$

Notice that in the illustrative pictures below the R–axis is vertical, whereas the δ–axis is horizontal: this is the traditional choice.

Denote by V_q the set of all points P_C, corresponding to $[n,k,d]_q$–codes with fixed q. Let U_q be the set of limit points of V_q.

In the latter definition, there is a subtlety. Logically, it might happen that one and the same code point corresponds to an infinite family of different codes, but is not a limit point. Then we would have a choice, whether to include such points to U_q automatically or not. However, we will show below (Theorem 2.10), that in fact two possible versions of definition lead to one and the same U_q.

0.3. Asymptotic bound. The main result about code domain is this: U_q *consists of all points in* $[0,1]^2$ *lying below the graph of a certain continuous decreasing function denoted* α_q*:*

$$U_q = \{(R,\delta)\,|\,R\le\alpha_q(\delta\}. \tag{0.2}$$

This curve is called *the asymptotic bound*. Surprisingly little is known about it: only various lower bounds, obtained using statistical estimates and explicit constructions of families of codes, and upper bounds, obtained by rather simple count.

In any case, this bound is the main theoretical result describing limitations imposed by the conflict between transmission rate and relative minimal distance.

0.4. Asymptotic bounds for structured codes. If we want to take into account limitations imposed by the feasibility of construction, encoding and decoding as well, we must restrict the set of considered codes, say, to a subset consisting of linear codes, or else polynomial time constructible/decodable codes etc. Linear codes produce the set of code points denoted V_q^{lin} and the set of its limit points denoted U_q^{lin}. The latter domain admits a description similar to (0.2), this time with another asymptotic bound α_q^{lin}. Clearly,

$$\alpha_q^{lin}(\delta) \le \alpha_q(\delta),$$

but whether this inequality is strict is seemingly unknown.

Adding the restriction of polynomial computability, we get in the same way asymptotic bounds $\alpha_q^{pol}(\delta)$ and $\alpha_q^{lin,pol}(\delta)$, which are continuous and decreasing and lie below the previous two bounds: see [ManVla] and [TsfaVla].

Proofs of (0.2) and its analogs are based upon a series of operations that allow one to obtain from a given code a series of codes with *worse* parameters: the so called *Spoiling Lemma(s).* They form the subject of the next section.

0.5. Asymptotic bounds as phase transitions. In view of (0.2), a picture of the closure of V_q would consist of the whole domain under the graph of α_q plus a cloud of isolated code points above it. In a sense, the best codes are (some) isolated ones: cf. our discussion in 2.5 and 2.6 below.

This picture reminds us e. g. of phase diagrams in physics, say, on the plane *(temperature, pressure)*, and alike. One of the goals of this paper is to elaborate on this analogy.

To this end, we give several interpretations of R and δ as "fractional dimensions", fractal and von Neumann's ones.

1. Spoiling Lemma

1.1. Code parameters reconsidered. For linear codes, k is always an integer. For general codes, this fails. One can define U_q using any one of the numbers k/n, $[k]/n$. As is easily seen, they provide *the same* asymptotic bound $R = \alpha_q(\delta)$: $(k_i/n_i, d_i/n_i)$ and $([k_i]/n_i, d_i/n_i)$ diverge or converge simultaneously and have the same limit. Working with both k and $[k]$, depending on the context, can be motivated as follows.

(i) k supplies the precise cardinality of C, and the precise transmission rate, but allows code points with irrational coordinates. This introduces unnecessary complications both in the study of computability properties of the code domains and in the statements of spoiling lemmas.

(ii) $[k]$ gives only estimates for $\#C$, but better serves spoiling. Moreover, in the eventual studies of computability properties of the graph $R = \alpha_q(\delta)$, it will be important to approximate it by points with rational coordinates, rather than logarithms.

Unless stated otherwise, we associate with an $[n,k,d]_q$–code C the code point $(R(C) := k/n, \delta(C) := d/n)$, and define the family V_q and the set U_q using these code points.

1.1.1. Spoiling operations. Having chosen a code $C \subset A^n$ and a pair (f,i), $f \in Map\,(C,A)$, $i \in \{1,\dots,n\}$, define three new codes:

$$C_1 =: C *_i f \subset A^{n+1} :$$

$$(a_1,\dots,a_{n+1}) \in C_1 \text{ iff } (a_1,\dots,a_{i-1},a_{i+1},\dots,a_n) \in C\,,$$
$$\text{and } a_i = f(a_1,\dots,a_{i-1},a_{i+1}\dots,a_n)\,. \tag{1.1}$$

$$C_2 =: C*_i \subset A^{n-1} :$$
$$(a_1,\dots,a_{n-1}) \in C_2 \text{ iff } \exists b \in A,\ (a_1,\dots,a_{i-1},b,a_{i+1},\dots,a_n) \in C. \tag{1.2}$$

$$C_3 =: C(a,i) \subset C \subset A^n : \qquad (a_1,\dots,a_n) \in C_3 \text{ iff } a_i = a. \tag{1.3}$$

In plain words: operation $*_i f$ inserts the letter $f(x)$ between the $(i-1)$–th and the i–th letters of each word $x \in C$; operation $*_i$ deletes the i–th letter of each word, i. e. projects the code to the remaining coordinates; and (a,i) collects those words of C that have a at the i–th place.

Assume now that C is linear.

Then $C *_i f$ remains linear, if $f : C \to A = \mathbf{F}_q$ is a linear function. Moreover, $C*_i$ is always linear. Finally, $C(a,i)*_i$ is also linear for any a.

These remarks will be used in order to imply that Corollary 1.2.1. below remain true if we restrict ourselves to linear codes.

1.2. Lemma. *If C is an $[n,k,d]_q$–code, then the codes obtained from it by application of one of these operations have the following parameters:*

(i) $C_1 = C *_i f$: $[n+1,k,d]_q$, if f is a constant function.

(i′) $C_1 = C *_i f$: $[n+1,k,d+1]_q$, if for each pair $x,y \in C$ with $d(x,y) = d$, we have $f(x) \neq f(y)$.

(ii) $C_2 = C*_i$: $[n-1, k, d]_q$, *if for each pair* $x, y \in C$ *with* $d(x, y) = d$, *these points have one and the same letter at the place* i.

Otherwise $[n-1, k, d-1]_q$.

(iii) $C_3 := C(a, i)$. *In this case, for each* i, *there exists such a letter* $a_i \in A$ *(perhaps, not unique) that*

$$\#C(a_i, i) \geq q^{k-1}. \tag{1.4}$$

Therefore, the code $C(a_i, i)*_i$ *will have parameters in the following range:*

$$[n-1,\ k-1 \leq k' < k,\ d' \geq d]_q. \tag{1.5}$$

Proof. The statements $(i), (i')$ and (ii) are straightforward. For (iii), remark that for any fixed i, C is the disjoint union of $C(a, i)$, $a \in A$. Hence

$$\sum_{a \in A} \#C(a, i) = q^k \tag{1.6}$$

and $\#A = q$ together imply (1.4) for at least of one of $C(a, i)$. Passing to $C(a_i, i)*_i$, we are deleting the i–th letter of all code words, which is common for all of them, so that the minimal distance does not change. But for subcodes of C it may be only d or larger.

1.2.1. Corollary (Numerical spoiling). *If there exists a code* C *with parameters* $[n, k, d]_q$, *then there exist also a code with the following parameters:*

(i) $[n+1, k, d]_q$ *(always).*

(ii) $[n-1, k, d-1]_q$ *(if* $n > 1, k > 0$.*)*

(iii) $[n-1, k-1 \leq k' < k, d]_q$ *(if* $n > 1,\ k > 1$).

The same remains true in the domain of linear codes.

Proof. Lemma 1.2 (i) provides the first statement.

In order to be able to use Lemma 1.2 (ii) for the second statement, we must find a pair of words at the distance d in C, that have different letters at some place i. This is always possible if $\#C \geq 2$, $n \geq 2$.

The case (iii) can be treated as follows.

If C can be represented in the form $C' *_i a$ where a denotes the constant function $x \mapsto a \in C$, then C' is an $[n-1, k, d]_q$–code. More generally, take the maximal projection of C (onto some coordinate quotient set A^m) that is injective on C and therefore preserves k, d. We will get an $[m, k, d]_q$–code with $n > m \geq 2$, because for $m = 1$ we must have $0 < k \leq 1$, the case that we have excluded in (iii). If we manage to worsen its parameters to $[m, k', d]_q$, $k - 1 \leq k' < k$, then afterwards using (i) several times, we will get an $[n-1, k', d]_q$–code.

Therefore, it remains to treat the case when C cannot be represented in the form $C' *_i a$. In this case, in the sum (1.6) there are at least two non–vanishing summands. Hence for the respective code $C(a, i)$ satisfying (1.4), we have also

$$q^{k-1} \leq \#C(a_i, i) < q^k. \tag{1.7}$$

Therefore

$$[k(C(a, i)*_i)] = [k] - 1. \tag{1.8}$$

It might happen that $d(C(a_i, i)) > d$. In this case we can apply to $C(a_i) * i$ several times (ii) and then several times (i).

1.3. Remark. In the next section, we will prove the existence of the asymptotic bound using only the numerical spoiling results of Corollary 1.2.1. Thus such a bound exists for any subclass of (structured) codes stable with respect to an appropriate family of spoiling operations, in particular, for linear codes. Computational feasibility of spoiled codes must in principle be checked separately, but it holds for usual formalizations of polynomial time computability.

2. Asymptotic bound: existence theorem and unsolved problems

2.1. Controlling cones. Let $P = (R_P, \delta_P)$ be a point of the square $[0,1]^2$ with $R_P + \delta_P < 1$. All points of U_q belong to this domain Δ.

For two points P, Q, denote by $[P, Q]$ the closed segment of the line $l(P, Q)$ connecting P and Q.

For $P \in \Delta$, consider two segments $[P, (1, 0)]$ and $[P, (0, 1)]$, The part of Δ, bounded by these two segments and the diagonal $R_P + \delta_P = 1$, will be called *the upper (controlling) cone of* P and denoted $C(P)_{up}$.

Extend $[P, (1, 0)]$ (resp. $[P, (0, 1)]$) from their common point P until their first intersection points with δ–axis (resp. R–axis). Then Δ will be broken into four parts: the upper cone $C(P)_{up}$, the *lower cone* $C(P)_{low}$ lying below the lines $l(P, (1, 0))$ and $l(P, (0, 1))$, *the left cone* $C(P)_l$ and *the right cone* $C(P)_r$. We agree to include into each cone two segments of its boundary issuing from P.

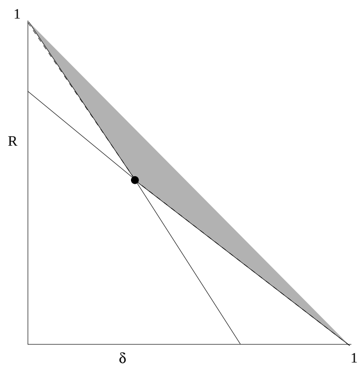

Fig. 1. *Controlling cones*

Let $P, Q \in \Delta$.

2.1.1. Lemma. *If $P \in U_q$, then $C(P)_{low} \subset U_q$.*

This follows from the Spoiling Lemma. In the proof, it is convenient to use the code points $([k]/n, d/n)$ rather than $(k/n, d/n)$.

In fact, if a sequence of code points $Q_i = ([k_i]/n_i, d_i/n_i)$ (q being fixed) tends to the limit point (R, δ), then the following statements are straightforward.

(a) $n_i \to \infty$.

(b) The boundaries of $C(Q_i)_{low}$ converge to the boundary of $C((R, \delta))_{low}$. Moreover, the boundaries of $C(Q_i)_{low}$ contain code points that become more and more dense when $n_i \to \infty$, namely $([k_i] - a)/n_i, d_i/n_i)$ and $([k_i]/n_i, (d_i - b)/n_i)$, $a, b = 1, 2, \ldots$ (Spoiling Lemma).

Thus, the whole boundary of $C((R, \delta))_{low}$ belongs to U_q.

(c) When a point Q moves, say, along the right boundary segment of $C((R, \delta))_{low}$, the left boundary segment of $C(Q)_{low}$ sweeps the whole $C((R, \delta))_{low}$.

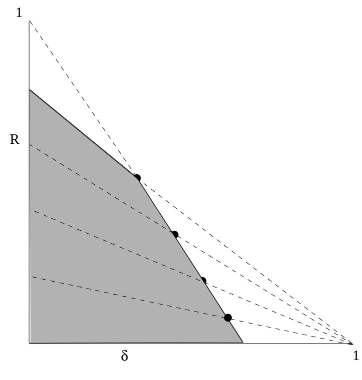

Fig. 2. *Code points on the lower cone boundary*

This completes the proof of the Lemma.

2.1.2. Lemma. *(i) If $P \in C(Q)_l$, then $Q \in C(P)_r$, and vice versa.*

(ii) If $P \in C(Q)_{low}$, then $Q \in C(P)_{up}$, and vice versa.

This is straightforward; a simple picture shows the reason.

2.1.3. Lemma. *If $P, Q \in \Gamma(\alpha_q)$ and $\delta_P < \delta_Q$, then $P \in C(Q)_l$, and therefore $Q \in C(P)_r$.*

Proof. In fact, otherwise P must be an inner point of $C(Q)_{low}$, (or the same with P, Q permuted). But no boundary point of U_q can lie in the lower cone of another boundary point.

2.1.4. Controlling quadrangles. Let $P, Q \in \Delta$, $\delta_P < \delta_Q$, and $P \in C(Q)_l$. Put

$$C(P, Q) := C(P)_R \cap C(P)_l.$$

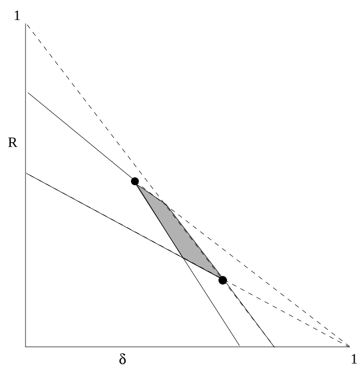

Fig. 3. *Controlling quadrangle*

When $P, Q \in \Gamma(\alpha_q)$ we will call $C(P, Q)$ the *controlling quadrangle* with vertices P, Q.

2.1.5. Lemma. *All points of $\Gamma(\alpha_q)$ between P and Q belong to $C(P, Q)$.*

This follows from Lemma 2.1.3.

These facts suffice to prove the following result ([Man], [ManVla]).

2.2. Theorem. *For each $\delta \in [0, 1]$, put*

$$\alpha_q(\delta) := \sup \{R_P \mid P = (R_P, \delta) \in U_q\}.$$

Then

(i) α_q *is a continuous decreasing function. Denote its graph by* $\Gamma(\alpha_q)$. *We have* $\alpha_q(0) = 1$, $\alpha_q(\delta) = 0$ *for* $\delta \in [(q-1)/q, 1]$.

(ii) U_q *consists of all points lying below or on* $\Gamma(\alpha_q)$. *It is the union of all lower cones of points of* $\Gamma(\alpha_q)$.

(iii) Each horizontal line $0 < R = const < 1$ *intersects* $\Gamma(\alpha_q)$ *at precisely one point, so that the* $\Gamma(\alpha_q)$ *is also the graph of the inverse function.*

The same statement remains true, if we restrict ourselves by a subclass of structured codes, for which Corollary 1.2.1 holds.

2.3. Corollary. *The curve* $\Gamma(\alpha_q)$ *(asymptotic bound) is almost everywhere differentiable.*

This follows from the fact that it is continuous and monotone (Lebesgue's theorem).

2.4. Problem. *(i) Is* $\Gamma(\alpha_q)$ *differentiable, or at least peacewise differentiable?*

(ii) Is this curve concave?

2.5. Isolated codes and excellent codes. Any code whose point lies strictly above $\Gamma(\alpha_q)$ is called *isolated* one. Consider the union W_q of lower cones of all isolated codes. This is a domain in Δ bounded from above by a piecewise linear curve, union of fragments of bounds of these lower cones containing their vertices. A code is called *excellent* one, if it is isolated and is the vertex of one of such fragments.

2.6. Problem. *(i) Describe (as many as possible) excellent codes.*

(ii) Are Reed–Solomon codes excellent in the class of linear, or even all codes?

Reed–Solomon codes are certainly isolated, because they lie on the Singleton boundary $R = 1 - \delta + 1/(q+1)$ which is higher than Plotkin's asymptotic bound

$$\alpha_q(\delta) \leq 1 - \delta - \frac{1}{q-1}\delta.$$

One easily sees that the set of isolated points is infinite, and that points $R = 1, \delta = 0$ and the segment $R = 0, (q-1)/q \leq \delta \leq 1$ are limit points for this set.

2.7. Problem. *Are there points on* $\Gamma(\alpha_q)$, $0 < R < 1$ *that are limit points of a sequence of isolated codes?*

2.8. Code domain and computability. The family V_q is a recursive subfamily of $\mathbf{Q}$: generating all codes and their code points, we get an enumeration of V_q. Let $W_q := \operatorname{supp} V_q$ be the set of all code points.

2.8.1. Question. *Is W_q a decidable set?*

2.8.2. Problem. *Are the following sets enumerable, or even decidable:*

(i) $\{(R(C),\delta(C))\,|\,R(C) < \alpha_q(\delta(C))\}$.

(ii) $\{(R(C),\delta(C))\,|\,R(C) \le \alpha_q(\delta(C))\}$.

(iii) $\{(R(C),\delta(C))\,|\,R(C) > \alpha_q(\delta(C))\}$.

(iv) $\{(R(C),\delta(C))\,|\,R(C) \ge \alpha_q(\delta(C))\}$.

2.9. Codes of finite and infinite multiplicity. Let (R,δ) be the code point of a code C. We will say that this point (and C itself) has the finite (resp. infinite) multiplicity, if the number of codes (up to isomorphism) corresponding to this point is finite (resp. infinite).

If C has parameters $[n,k,d]_q$, then codes with the same code point have parameters $[an,ak,ad]_q$, $a \in \mathbf{Q}_+^*$. Clearly, finite (resp. infinite) multiplicity of C can be inferred by looking at whether there exist finitely or infinitely many $a \in \mathbf{Q}_+^*$ such that an $[an,ak,ad]_q$-code exists for such a. Moreover, from the proof below one sees that one can restrict oneself by looking only at integer a.

2.10. Theorem. *Assume that the code point of C does not lie on the asymptotic bound. Then it has finite multiplicity iff it is isolated.*

Proof. If C is of infinite multiplicity, it cannot be isolated. In fact, spoiling all codes with parameters $[an,ak,ad]_q$, we get a dense set of points on the boundary of the lower cone of the respective point.

Conversely, let an $[n,k,d]_q$–code C lie below the asymptotic bound. Then there exist $[N,K,D]_q$–codes with arbitrarily large N,K,D satisfying the conditions

$$\frac{K}{N} > \frac{k}{n}, \quad \frac{D}{N} > \frac{d}{n}. \tag{2.1}$$

Slightly enlarging N by spoiling, we may achieve $N = an$, with $a \in \mathbf{N}$. Let

$$K = ak' + a_1,\ 0 \le a_1 < a,\ k' \in \mathbf{N},$$

$$D = ad' + a_2,\ 0 \le a_2 < a,\ d' \in \mathbf{N},$$

In view of (2.1), we have

$$ak' + a_1 > ak, \quad ad' + a_2 > ad.$$

To complete the proof, it remains to reduce the parameters K, D by spoiling, and get an $[an, ak, ad]_q$–code; a can be arbitrarily large.

2.11. Question. *Can one find a recursive function $b(n,k,d,q)$ such that if an $[n,k,d]_q$–code is isolated, and $a > b(n,k,d,q)$, there is no code with parameters $[an, ak, ad]_q$?*

3. Code fractals: rate and relative minimum distance as Hausdorff dimensions

3.1. Code rate and the Hausdorff dimension. In this subsection we will show that the rate R of a code C has a simple geometric interpretation as the Hausdorff dimension of a Sierpinski fractal naturally associated to the code.

We start with choosing a bijection of the initial alphabet A with q–ary digits $\{0, 1, \dots, q-1\}$. Intermediary constructions will depend on it, but basic statements will not. For the time being, we will simply identify A with digits.

The rational numbers with denominators q^n, $n \geq 0$, admit two different infinite q–ary expansions. Therefore we will exclude them, and put

$$(0,1)_q := [0,1] \setminus \{m/q^n \mid m, n \in \mathbf{Z}\} \tag{3.1}$$

The remaining points of the cube $x = (x_1, \dots, x_n) \in (0,1)_q^n$ can be identified with $(\infty \times n)$–matrices with entries in A: the k–th column of this matrix consists of the consecutive digits of the q–ary decomposition of x_k.

Now, for a code $C \subset A^n$, denote by $S_C \subset (0,1)_q^n$ the subset consisting of those points x, for which each line of the respective matrix belongs to C. This is a Sierpinski fractal.

3.2. Proposition. *The Hausdorff dimension $s := \dim_H(S_C)$ equals to the rate $R = R(C)$.*

Proof. S_C is covered by $\#C = q^k$ cubes of size q^{-1}, consisting of such points in $(0,1)^n$ that the first line of their coordinate matrix belongs to C. Inside each such small cube lies a copy of S_C scaled by q^{-1}. This self–similarity structure shows that s is the solution to the equation $(\#C)q^{-ns} = 1$ (see §9.2 of [Fal]). Hence

$$\dim_H(S_C) = \frac{\log(\#C)}{n \log q} = \frac{k}{n} = R. \tag{3.2}$$

Remark. Several different notions of fractal dimension (Hausdorff dimension, box counting dimension, and scaling dimension) agree for S_C, hence the Hausdorff dimension can be computed from the simple self–similarity equation.

3.3. Relative minimum distance and the Hausdorff dimension. The most straightforward way to connect the relative minimum distance of a code C with Hausdorff dimension is to consider intersections of S_C with l–dimensional linear subspaces $\pi = \pi^l$ that are translates of intersections of coordinate hyperplanes in $\mathbf{R}^n$, that is, are given by the equations $x_i = x_i^0$ for some $i = i_1, \dots, i_{n-l}$.

3.3.1. Proposition. *In this notation, we have:*

(i) If $l < d$, then $S_C \cap \pi$ is empty.

(ii) If $l \geq d$, then $S_C \cap \pi$ has positive Hausdorff dimension:

$$\dim_H(S_C \cap \pi) = \frac{\log \#(C \cap \pi)}{l \log q} > 0. \tag{3.3}$$

Proof. We will embed $C \subset A^n$ in $\mathbf{R}^n$ by sending $(x_1, \dots, x_n)$ to $(x_1/q, \dots, x_n/q)$. (Notice that all these points will lie in $[0,1]^n$, but outside of $(0,1)_q^n$.)

Then no two points of C will lie in one and the same l–dimensional π, if $n - l \geq n - d + 1$, because at least d of their coordinates are pairwise distinct. On the other hand, if $n - l \leq n - d$, then one can find π containing at least two points of C.

In terms of the iterative construction of the fractal S_C, this means the following. For a given π with $l \leq d - 1$, if the intersection $C \cap \pi$ is non–empty it must consist of a single point. Thus, at the first step of the construction of $S_C \cap \pi$ we must replace the single cube $(0,1)_q^n \cap \pi$ with a single copy of a scaled cube of volume q^{-l}, and then successively iterate the same procedure. This will produces a family of nested open cubes of volumes q^{-lN}. Their intersection is clearly empty.

When $l \geq d$, one can choose $\pi = \pi^d$ for which $C \cap \pi$ contains at least two points. Then the induced iterative construction of the set $S_C \cap \pi$ starts by replacing the cube $Q^d = Q^n \cap \pi$ with $\#(C \cap \pi)$ copies of the same cube scaled down to have volume q^{-d}. The construction is then iterated inside all the resulting $\#(C \cap \pi)$ cubes, so that one obtains a set of Hausdorff dimension $s = \dim_H(S_C \cap \pi)$ which is a solution to the equation $\#(C \cap \pi) \cdot q^{-ls} = 1$. Thus

$$\dim_H(S_C \cap \pi) = \frac{\log \#(C \cap \pi)}{l \log q} > 0.$$

This completes the proof.

One can refine this construction by associating a fractal set S_π to each subspace π as above. Namely, define S_π as the set of points of $(0,1)^n_q$ whose matrices have all rows in π.

3.4. Proposition. *The Hausdorff dimension of S_π is*

$$\dim_H S_\pi = \frac{l}{n}. \tag{3.4}$$

In particular, for $l = d$ one has $\dim_H S_\pi = \delta$.

Proof. The argument is similar to the one in the previous proof. We construct S_π by subdividing, at the first step, the cube $[0,1]^n$ into q^n cubes of volume q^{-n} and of these we keep only those that correspond to points whose first digit of the n-coordinates, in the q-ary expansion define a point $(x_{11}, \dots, x_{1n}) \in \pi \cap A^n$. We have $\#(\pi \cap A^n) = q^l$, hence at the first step we replace Q^n by q^l cubes of volume q^{-n}. The procedure is then iterated on each of these. Thus, the Hausdorff dimension of S_π is the number s satisfying $q^l q^{-ns} = 1$, i. e. (3.4).

One can now use S_π in place of π, to make the roles of rate and minimal relative distance more symmetric in the Hausdorff context. Namely, we obtain,

3.5. Proposition. *We have*

$$\dim_H(S_C \cap S_\pi) = \frac{\log \#(C \cap \pi)}{n \log q} \tag{3.5}$$

In particular, for all $l \leq d-1$, the set $S_C \cap S_\pi$ is empty.

For $l \geq d$, there exists a subspace π^l for which $\dim_H(S_C \cap S_\pi) > 0$ so that $S_C \cap S_\pi$ is a genuine fractal set.

Proof. Again, the argument is similar to the one we have already used.

The iterative construction of $S_C \cap S_\pi$ replaces the initial unit cube $[0,1]^n$ with $\#(C \cap \pi)$ cubes of volume q^{-n} given by points with first row $(x_{11}, \dots, x_{1n}) \in C \cap \pi$. The same procedure is then iterated on each of these smaller cubes. Thus, the Hausdorff dimension is given by the self-similarity condition $\#(C \cap \pi) q^{-ns} = 1$, which shows (3.5).

The same argument as above then shows that, for all $l \leq d-1$ one has $\#(C \cap \pi) = 1$, if $C \cap \pi$ is non-empty, while for $l \geq d$ there exists a choice of π for which

$\#(C \cap \pi) \geq 2$. This shows that once again d is the threshold value for which there exists a choice of $\pi \in \Pi_d$ for which $\dim_H(S_C \cap S_\pi) > 0$.

4. Operator algebras of codes.

4.1. Finitely generated Toeplitz–Cuntz algebras. We introduce a class of C^*–algebras related to codes. Starting with an arbitrary finite set D, we associate to it Toeplitz and Cuntz algebras, as in [Cu1], [Fow].

4.1.1. Definition *(i) The Toeplitz–Cuntz algebra TO_D is the universal unital C^*–algebra generated by a distinguished family of isometries T_d, $d \in D$, with mutually orthogonal ranges.*

(ii) The Cuntz algebra O_D is the universal unital C^*–algebra generated by a distinguished family of isometries S_d, $d \in D$, with mutually orthogonal ranges, and satisfying the condition

$$\sum_{a \in D} S_d S_d^* = 1. \tag{4.1}$$

Notice that $T_d T_d^*$ form pairwise orthogonal projections, so that operator

$$P_D := \sum_{a \in D} T_d T_d^* \in TO_D$$

is a projector. But it is not identical.

From the definition it follows that the canonical morphism $TO_D \to O_D$: $T_d \mapsto S_d$ generates the exact sequence

$$0 \to J_D \to TO_D \to O_D \to 0,$$

where J_D is the ideal generated by $1 - P_D$. The ideal J_D is isomorphic to the algebra of compact operators $\mathcal{K}$.

4.1.2. Functoriality with respect to D. The Toeplitz–Cuntz algebras TO_D are functorial with respect to arbitrary *injective* maps $f : D \to D'$: the respective morphism maps T_d to $T_{f(d)}$.

The Cuntz algebras are functorial only with respect to *bijections*: any bijection $f : D \to D'$ generates an isomorphism $O_D \to O_{D'}$ so that isomorphism class of O_D depends only on $\#D$. The algebra $O_{\{1,\dots N\}}$ is often denoted simply O_N.

Below we will consider, in particular, TO_C and O_C for codes C, including codes A^n. The last remark allows us to canonically identify versions of O_C that arise, for example, from different bijections $A \to \{0, \dots, q-1\}$, as in 3.1 where they were used for the construction of fractals S_C.

Functoriality of TO_D with respect to injections allows one to define the algebra $TO_\infty := TO_{\{1,2,\dots,\dots\}}$, see e.g. [Fow], identified with the algebra O_∞ considered by Cuntz in [Cu1] and treated separately there.

4.1.3. Fractals and algebras. In order to connect Toeplitz–Cuntz and Cuntz algebras TO_C, O_C with fractals S_C, it is convenient to introduce two other topological spaces closely related to S_C.

We will denote by $\bar{S}_C$ the closure of the set S_C inside the cube $[0,1]^n$, after identifying points of S_C with n-tuples of irrational points in $[0,1]$ written in their q–ary expansion. The set $\bar{S}_C$ is also a fractal of the same Hausdorff dimension as S_C, which now includes also the rational points with q-ary digits in C. It is a topological (metric) space in the induced topology from $[0,1]^n$.

We also consider the third space $\hat{S}_C$. It is a compact Hausdorff space, which maps surjectively to $\bar{S}_C$, one-to-one on S_C and two-to-one on the points of $\bar{S}_C \smallsetminus S_C$. By [Cu1] one knows that $\hat{S}_C$ is the spectrum of the maximal abelian subalgebra of the Cuntz algebra O_C.

$\hat{S}_C$ can be identified with the set of all infinite words $x = x_1 x_2 \cdots x_m \cdots$ with letters $x_i \in C$. Using the matrix language of 3.1, we can say that points of $\hat{C}$ corresponds to all (∞, n)–matrices whose line belong to C. The set S_C is dense in $\hat{S}_C$ as the subset of non-periodic sequences.

The map $\hat{S}_C \to \bar{S}_C$ identifies coordinatewise the two q-ary expansions of rational points with q–denominators in $\bar{S}_C$. The sets $\hat{S}_C$, $\bar{S}_C$ and S_C only differ on sets of Hausdorff measure zero, so for the purpose of measure theoretic considerations we often do not need to distinguish between them.

One can consider the abelian C^*-algebra $\mathcal{A}(\hat{S}_C)$ generated by the characteristic functions $\chi_{\hat{S}_C(w)}$, where $w = a_1 \cdots a_m$ runs over finite words with letters a_i in C, and $\hat{S}_C(w)$ denotes the subset of infinite words $x \in \hat{S}_C$ that start with the finite word w. This algebra is isomorphic to the maximal abelian subalgebra of O_C. In fact, these characteristic functions can be identified with the range projections $P_w = S_w S_w^* = S_{a_1} \cdots S_{a_m} S_{a_m}^* \cdots S_{a_1}^*$ in O_C. We also denote by $T\mathcal{A}(C)$ the abelian subalgebra of TO_C generated by the range projections $T_w T_w^*$, and which maps to $\mathcal{A}(\hat{S}_C)$ in the quotient algebra O_C.

Notice that, for an injective map $f : C \to C'$, the induced map $T_f : TO_C \hookrightarrow TO_{C'}$ induces also an embedding $T_f : \mathcal{A}(C) \hookrightarrow \mathcal{A}(C')$ of the respective abelian subalgebras:

$$T_w T_w^* \mapsto T_{f(w)} T_{f(w)}^* := T_{f(a_1)} \cdots T_{f(a_m)} T_{f(a_m)}^* \cdots T_{f(a_1)}^*.$$

For the sets $\hat{S}_C$ and the abelian algebras $\mathcal{A}(\hat{S}_C)$, one also has a functoriality in the opposite direction for more general maps $f : C \to C'$ of codes that are not necessarily injective. Namely, such a map induces a map $\hat{S}_C \to \hat{S}_{C'}$ that sends an infinite sequence $x = a_1 a_2 \cdots a_m \cdots$ with $a_i \in C$ to the infinite sequence $f(a_1) f(a_2) \cdots f(a_m) \cdots$ in $\hat{S}_{C'}$. Since the basis for the topology on $\hat{S}_C$ is given by the cylinder sets $\hat{S}_C(w)$, the map constructed in this way is continuous. This gives an algebra homomorphism $\mathcal{A}(\hat{S}_{C'}) \to \mathcal{A}(\hat{S}_C)$.

4.2. Representations of Cuntz algebras associated to S_C. In the following let us denote by $\sigma : S_C \to S_C$ the map that deletes the first row of the coordinate matrix, shifting to the left the remaining q-adic digits of the coordinates,

$$\sigma(x) = (x_{12} \ldots x_{1k} \ldots ; x_{22} \cdots x_{2k} \ldots ; \ldots ; x_{n2} \cdots x_{nk} \cdots) \tag{4.2}$$

for $x = (x_{11} x_{12} \cdots x_{1k} \cdots ; x_{21} x_{22} \cdots x_{2k} \cdots ; x_{n1} x_{n2} \cdots x_{nk} \cdots)$ in S_C, that is, shifting upward the remaining rows of the $\infty \times n$-matrix. For $a = (a_1, \ldots , a_n) \in C \subset A^n$, let σ_a denote the map adding a as the first row of the coordinate matrix

$$\sigma_a(x) = (a_1 x_{11} x_{12} \ldots x_{1k} \ldots ; a_2 x_{21} x_{22} \ldots x_{2k} \ldots ; \ldots ; a_n x_{n1} x_{n2} \cdots x_{nk} \cdots). \tag{4.3}$$

Since $a \in C$, (4.3) maps S_C to itself. These maps are partial inverses of the shift (4.2). In fact, if we denote by $R_a \subset S_C$ the range $R_a = \sigma_a(S_C)$, then on R_a one has $\sigma_a \sigma(x) = x$, while for all $x \in S_C$ one has $\sigma \sigma_a(x) = x$. We also introduce the notation

$$\Phi_a(x) = \frac{d\mu \circ \sigma_a}{d\mu}, \tag{4.4}$$

for the Radon-Nikodym derivative of the Hausdorff measure μ composed with the map σ_a.

Since the maps σ_a act on S_C by

$$\sigma_a(x_1, \ldots , x_n) = \left(\frac{x_1 + a_1}{q}, \ldots , \frac{x_n + a_n}{q} \right), \tag{4.5}$$

the Radon-Nikodym derivative Φ_a of (4.4), with μ the Hausdorff measure of dimension $s = \dim_H(S_C)$, is constant

$$\Phi_a(x) = \frac{d\mu \circ \sigma_a}{d\mu} = q^{-ns} = q^{-k}. \tag{4.6}$$

4.2.1. Proposition. *The operators*

$$(S_a f)(x) = \chi_{R_a}(x)\Phi_a(\sigma(x))^{-1/2} f(\sigma(x)) \tag{4.7}$$

determine a representation of the algebra O_C on the Hilbert space $L^2(S_C, \mu)$.

Proof. The adjoint of (4.7) in the L^2 inner product $\langle\, , \rangle$ is of the form

$$(S_a^* f)(x) = \Phi_a(x)^{1/2} f(\sigma_a(x)), \tag{4.8}$$

therefore $S_a S_a^* = P_a$, where P_a is the projection given by multiplication by the characteristic function χ_{R_a}, so that one obtains $\sum_a S_a S_a^* = 1$. Moreover, $S_a^* S_a = 1$, so that one obtains a representation of the C^*-algebra O_C.

Changing the identification of abstract code letters with q-ary digits corresponds to an action of the symmetry group Σ_q. The main invariants of codes like k and d only depend on the equivalence class under this action.

4.2.2. Proposition. *The action of the group Σ_q induces a unitary equivalence of the representations of the Cuntz algebras and a measure preserving homeomorphism of the limit sets.*

Proof. Suppose given an element $\gamma \in \Sigma_q$ and let $C' = \gamma(C)$ be the equivalent code obtained from C by the action of γ. The element γ induces a map $\gamma : S_C \to S_{C'}$ by

$$x = x_1 x_2 \cdots x_k \cdots \mapsto \gamma(x) = \gamma(x_1)\gamma(x_2)\cdots\gamma(x_k)\cdots .$$

This map is a homeomorphism. In fact, it is a bijection since $\gamma : C \to C'$ is a bijection, and it is continuous since the preimage of a clopen set $S_{C'}(w')$ of all words in $S_{C'}$ starting with a given finite word w' consists of the clopen set $S_C(w)$ with $w = \gamma^{-1}(w')$. Since both S_C and $S_{C'}$ are compact and Hausdorff, the map is a homeomorphism. It is measure preserving since the measure of the sets $S_C(w)$ is uniform in the words w of fixed length,

$$\mu(S_C(w)) = q^{-kr}, \quad \text{for all} \quad w = w_1, \dots, w_r, \ \ w_i \in C,$$

so the measure is preserved in permutations of coordinates.

Thus, the action of $\gamma : S_C \to S_{C'}$ determines a unitary equivalence $U_\gamma : L^2(S_{C'}, \mu) \to L^2(S_C, \mu)$, and a representation of the algebra O_C on $L^2(S_{C'}, \mu)$ generated by the operators $S'_a = U^*_\gamma S_a U_\gamma$. This completes the proof.

We have seen that, more abstractly, we can identify $\hat{S}_C$ with the spectrum of the maximal abelian subalgebra of the algebra O_C generated by the range projections $S_w S_w^*$, for words w of finite length. One can see in this way directly that the action of Σ_q induces homeomorphisms of these sets. The uniform distribution of the measure implies that these are measure preserving.

4.3. Perron–Frobenius and Ruelle operators. Consider again the shift map $\sigma : S_C \to S_C$ defined in (4.2). The Perron–Frobenius operator $\mathcal{P}_\sigma$ is the adjoint of composition by σ, namely

$$\langle h \circ \sigma, f \rangle = \langle h, \mathcal{P}_\sigma f \rangle. \tag{4.9}$$

4.3.1. Lemma. *The Perron–Frobenius operator $\mathcal{P}_\sigma$ is of the form*

$$\mathcal{P}_\sigma = q^{-k/2} \sum_{a \in C} S_a^*. \tag{4.10}$$

Proof. We have

$$\int_{S_C} \overline{h \circ \sigma} \cdot f \, d\mu = \sum_a \int_{R_a} \overline{h \circ \sigma} \cdot f \, d\mu = \sum_a \int_{S_C} \overline{h} \cdot f \circ \sigma_a \cdot \Phi_a \, d\mu,$$

with $R_a = \sigma_a(S_C)$, so that we have

$$\mathcal{P}_\sigma f = \sum_a \Phi_a f \circ \sigma_a = \sum_a \Phi_a^{1/2} S_a^* f = q^{-k/2} \sum_a S_a^* f.$$

This gives (4.10) and completes the proof.

Remark. A modified version of the Perron–Frobenius operator which is also useful to consider is the Ruelle transfer operator for the shift map $\sigma : S_C \to S_C$ with a potential function $W : S_C \to \mathbf{C}$. One usually assumes that the potential takes non-negative real values. The Ruelle transfer operator $\mathcal{R}_{\sigma,W}$ is then defined as

$$\mathcal{R}_{\sigma,W} f(x) = \sum_{y : \sigma(y) = x} W(y) \, f(y). \tag{4.11}$$

For a real valued potential, the operator $\mathcal{R}_{\sigma,W}$ is also obtained as the adjoint of $h \mapsto q^k\, W \cdot h \circ \sigma$,

$$\langle q^k\, W \cdot h \circ \sigma, f \rangle = \langle h, \mathcal{R}_{\sigma,W} f \rangle,$$

hence it can be regarded as a generalization of the Perron–Frobenius operator. The Ruelle and Perron–Frobenius operators are related to the existence of invariant measures on S_C and of KMS states for the algebra O_C, with respect to time evolutions related to the potential W.

4.4. Time evolution and KMS states. We recall some well known facts about KMS states on the Cuntz algebras, see for instance [KiKu], [KuRe].

Given a set of real numbers $\{\lambda_1, \ldots, \lambda_N\}$ there is a time evolution on the Cuntz algebra O_N which is completely determined by setting

$$\sigma_t(S_k) = e^{it\lambda_k} S_k. \tag{4.12}$$

Recall that a KMS state at inverse temperature β on a C^*-algebra $\mathcal{B}$ with a time evolution σ_t is a state $\varphi : \mathcal{B} \to \mathbf{C}$, such that for each $a, b \in \mathcal{B}$ there exists a holomorphic function F_{ab} on the strip $0 < \Im(z) < \beta$, which extends continuously to the boundary of the strip and satisfies

$$F_{ab}(t) = \varphi(a\sigma_t(b)), \quad \text{and} \quad F_{ab}(t + i\beta) = \varphi(\sigma_t(a)b).$$

4.4.1. Proposition. *For the time evolution (4.12) on the Cuntz algebra O_N, there exists a unique KMS state at inverse temperature $\beta > 0$ if and only if β satisfies*

$$\sum_{k=1}^{N} e^{-\beta\lambda_k} = 1. \tag{4.13}$$

Proof. If $\{\lambda_1, \ldots, \lambda_N\}$ and a β satisfy (4.13), then the λ_k are all positive and define β uniquely.

As in [KuRe], one uses the Ruelle transfer operator on the set X of infinite sequences in an alphabet on N-letters. For a potential $W(x) = e^{-\beta\lambda_{x_1}}$, where $x = x_1 x_2 \cdots x_n \cdots$, one finds that the constant function 1 is a fixed point of $\mathcal{R}_{\sigma,W}$,

$$\mathcal{R}_{\sigma,W} 1 = (\sum_k e^{-\beta\lambda_k}) 1,$$

hence dually there is a probability measure $\mu_{\lambda,\beta}$ on X which is fixed by the dual operator, $\mathcal{R}^*_{\sigma,W}\mu_{\lambda,\beta} = \mu_{\lambda,\beta}$. This is a measure satisfying a self-similarity condition on X. In fact, one has

$$\mathcal{R}^*_{\sigma,W}\mu_{\lambda,\beta} = W\frac{d\mu_{\lambda,\beta}\circ\sigma}{d\mu_{\lambda,\beta}}\mu_{\lambda,\beta},$$

so that $\mathcal{R}^*_{\sigma,W}\mu_{\lambda,\beta} = \mu_{\lambda,\beta}$ implies that

$$\frac{d\mu_{\lambda,\beta}\circ\sigma_k}{d\mu_{\lambda,\beta}} = e^{-\lambda_k\beta},$$

and hence $\mu_{\lambda,\beta}$ satisfies the self-similarity condition

$$\mu_{\lambda,\beta} = \sum_{k=1}^{N} e^{-\lambda_k\beta}\mu_{\lambda,\beta}\circ\sigma_k^{-1}.$$

The measure $\mu_{\lambda,\beta}$ is determined by the values $\mu_{\lambda,\beta}(R_k) = e^{-\beta\lambda_k}$, since then the value on a clopen set $X(w) \subset X$ of all infinite works starting with a given finite word w of length r is given by

$$\mu_{\lambda,\beta}(X(w)) = \int_X \frac{d\mu_{\lambda,\beta}\circ\sigma^\ell}{d\mu_{\lambda,\beta}}d\mu_{\lambda,\beta} = e^{-\lambda_{w_1}\beta}\cdots e^{-\lambda_{w_r}\beta},$$

which is consistent with $\mu_{\lambda,\beta}(X(w)) = \sum_{k=1}^N \mu_{\lambda,\beta}(X(wk))$.

By the spectral theory of the operator $\mathcal{R}_{\sigma,W}$ one knows, see [KuRe], that the fixed points $\mathcal{R}_{\sigma,W}1 = 1$ and $\mathcal{R}^*_{\sigma,W}\mu_{\lambda,\beta} = \mu_{\lambda,\beta}$ are unique. This gives then a unique KMS state on O_N at inverse temperature the unique β satisfying (4.13), which is given by integration with respect to the measure $\mu_{\lambda,\beta}$ composed with a continuous linear projection $\Phi : O_N \to C(X)$.

The latter is defined as follows: $\Phi(S_w S^*_{w'}) = 0$, if $w \neq w'$, and $\chi_{X(w)}$ otherwise, where w and w' are finite words in the alphabet on N letters. The state

$$\varphi_\beta(S_w S^*_{w'}) = \int \Phi(S_w S^*_{w'})d\mu_{\lambda,\beta} = \delta_{w,w'}e^{-\beta\lambda_{w_1}}\cdots e^{-\beta\lambda_{w_r}}, \tag{4.14}$$

for w of length r, is a KMS state on O_N at inverse temperature β. One sees that it satisfies the KMS condition since it suffices to see that $\varphi_\beta(S_w S^*_w) = \varphi_\beta(\sigma_{i\beta}(S^*_{w'})S_w)$.

It suffices then to check the latter identity for a single generator, and use the relations in the algebra to obtain the general case. One has $\varphi(S_k S_k^*) = e^{-\beta\lambda_k} = \varphi_\beta(e^{-\beta\lambda_k} S_k^* S_k) = \varphi_\beta(\sigma_{i\beta}(S_k^*)S_k)$.

This completes the proof.

Remark. Notice that (4.13) can be interpreted as the equation that computes the Hausdorff dimension of a self-similar set where the recursive construction replaces at the first step a set of measure one with N copies of itself, each scaled by a factor $e^{-\lambda_k}$ and then iterates the procedure.

In particular, in the main example we are considering here, of the Sierpinski fractal $S_C \subset Q^n$, the Hausdorff measure μ_s on S_C with parameter $s = \dim_H(S_C) = k/n$ is a self-similar measure as above, and it corresponds to the unique KMS state on the algebra O_C at inverse temperature $\beta = \dim_H(S_C) = k/n$, for the time evolution

$$\sigma_t(S_a) = q^{-itn} S_a, \tag{4.15}$$

for all $a \in C$. In fact, in this case the measure satisfies $\mu_s(R_a) = q^{-ns} = q^{-k}$ for all $a \in C$. Thus, the KMS state $\varphi_{k/n}$ takes values $\varphi_{k/n}(S_w S_w^*) = q^{-kr}$ for a word $w = w_1 \cdots w_r$, with $w_i = (a_{i1}, \ldots, a_{in}) \in C$.

4.5. KMS states and dual traces. Let Π_ℓ be the set of translates of ℓ-dimensional intersections of $n - \ell$ coordinate hyperplanes. To each $\pi \in \Pi_\ell$ we associate a projection in the algebra O_C, by taking

$$P_\pi = \sum_{a \in C \cap \pi} S_a S_a^*. \tag{4.16}$$

The value of the unique KMS state of O_C at this projection is

$$\varphi_{k/n}(P_\pi) = q^{-k} \cdot \#(C \cap \pi) = q^{\ell s - k}, \tag{4.17}$$

where $s = \dim_H(S_C \cap \pi)$.

Consider then the algebra obtained by compressing O_C with the projection P_π, that is, the algebra generated by the elements $S_{\pi(a)} := P_\pi S_a P_\pi$. These are non trivial when $a \in C \cap \pi$, in which case $S_{\pi(a)} = S_a$, and zero otherwise, and they satisfy the relations $S_{\pi(a)}^* S_{\pi(a)} = 1$, when $S_{\pi(a)}$ is non-trivial, and

$$\sum_a S_{\pi(a)} S_{\pi(a)}^* = P_\pi.$$

Thus, the algebra obtained by compressing with the projection P_π is a Toeplitz algebra $TO_{C\cap\pi}$.

The induced action on the Hilbert space $L^2(S_C \cap \pi, \mu_s)$ of the algebra $TO_{C\cap\pi}$ obtained as above descends to the quotient as a representation of $O_{C\cap\pi}$.

On the algebra $O_{C\cap\pi}$ generated by the S_a with $a \in C \cap \pi$, one can similarly consider a time evolution of the form (4.12), with the λ_a given by

$$\lambda_a = -\log \mu_s(R_a), \tag{4.18}$$

where μ_s is the Hausdorff measure in dimension $s = \dim_H(S_C \cap \pi)$. Then one has a unique KMS state on $O_{C\cap\pi}$ at inverse temperature $\beta = \dim_H(S_C \cap \pi)$, which is determined by integration in this Hausdorff measure.

In the following we look for a reinterpretation of the Hausdorff dimensions considered above in terms of von Neumann dimensions. To this purpose, we need to consider a type II von Neumann algebra. As we will see below, there are two ways to associate a type II algebra to the type III algebras O_C that we considered above. The first is passing to the dual system by taking the crossed product by the time evolution and the second is considering the fixed point algebra in the weak closure of the GNS representation. We finish this subsection by showing that the first method may not give the needed projections due to the projectionless nature of the resulting algebra. We then consider the second possibility in the next subsection, and see that one can obtain in that way the desired interpretation as von Neumann dimensions.

It is well known from [Co2] that, to a C^*-algebra $\mathcal{B}$ with time evolution σ_t, one can associate a dual system $(\hat{\mathcal{B}}, \theta)$, where $\hat{\mathcal{B}} = \mathcal{B} \rtimes_\sigma \mathbf{R}$ endowed with a dual scaling action of $\mathbf{R}_+^*$ of the form $\theta_\lambda(\int_{\mathbf{R}} a(t) U_t dt) = \int_{\mathbf{R}} \lambda^{it} a(t) U_t dt$. A KMS state φ_β at inverse temperature β on $(\mathcal{B}, \sigma)$ determines a dual trace τ_β on $\hat{\mathcal{B}}$, with the scaling condition

$$\tau_\beta \circ \theta_\lambda = \lambda^{-\beta} \tau_\beta. \tag{4.19}$$

The dual algebra $\hat{\mathcal{B}}$ is generated by elements of the form $\rho(f)a$, with $a \in \mathcal{B}$ and $f \in L^1(\mathbf{R})$ and with $\rho(f) = \int_{\mathbf{R}} f(t) U_t\, dt$. The dual trace is then of the form

$$\tau_\beta(\rho(f)a) = \varphi_\beta(a) \int_{\mathbf{R}} \hat{f}(s) e^{-\beta s} ds,$$

where $\hat{f}$ is the Fourier transform of $f \in L^1(\mathbf{R})$. Equivalently, for elements of the form $f \in L^1(\mathbf{R}, \mathcal{B})$ one has $\tau_\beta(f) = \int_{\mathbf{R}} \varphi_\beta(\hat{f}(s)) e^{-\beta s} ds$.

If the trace τ_β dual to a KMS state φ_β is a faithful trace, then, as observed in [Co], p.586, any projection P in $\hat{\mathcal{A}}$ is homotopic to $\theta_1(P)$ so that one should have $\tau_\beta(\theta_1(P)) = \tau_\beta(P)$, but the scaling property (4.19) implies that this is also $\tau_\beta(\theta_1(P)) = \lambda^{-\beta}\tau_\beta(P)$ so that one has $\tau(P) = 0$, which by faithfulness gives $P = 0$.

4.6. Hausdorff dimensions and von Neumann dimensions. We show that one can express the Hausdorff dimensions of the sets $S_C \cap \pi$ in terms of von Neumann dimensions of projections associated to the linear spaces π in the hyperfinite type II_1 factor.

4.6.1. Proposition. *Let $C \subset A^n$ be a code with $\#C = q^k$ and let $\pi \in \Pi_\ell$ be an ℓ-dimensional linear space as above, to which we associate the set $S_C \cap \pi$. To these data one can associate a projection P_π in the hyperfinite type II_1 factor with von Neumann trace τ, so that the von Neumann dimension* $\mathrm{Dim}\,(\pi) := \tau(P_\pi)$ *is related to the Hausdorff dimension of $S_C \cap \pi$ by*

$$\dim_H(S_C \cap \pi) = \frac{k + \log_q \mathrm{Dim}(\pi)}{\ell}, \qquad (4.20a)$$

$$\dim_H(S_C \cap S_\pi) = \frac{k + \log_q \mathrm{Dim}(\pi)}{n}. \qquad (4.20b)$$

Proof. When we consider as above the algebra O_C with the time evolution σ_t of (4.15), we can consider the spectral subspaces of the time evolution, namely

$$\mathcal{F}_\lambda = \{X \in O_C \mid \sigma_t(X) = \lambda X\}. \qquad (4.21)$$

In particular, $\mathcal{F}_0 \subset O_C$ is the fixed point subalgebra of the time evolution. This is generated linearly by elements of the form $S_w S_{w'}^*$, for words $w = w_1 \cdots w_r$ and $w' = w'_1 \cdots w'_r$ word of equal length in elements $w_j, w'_j \in C$. The fixed point algebra $\mathcal{F}_0$ contains the subalgebra $\mathcal{A}(\hat{S}_C)$ identified with the algebra generated by the $S_w S_w^*$. One has a conditional expectation $\Phi : O_C \to \mathcal{F}_0$ given by

$$\Phi(X) = \int_0^{2\pi/n \log q} \sigma_t(X) dt \qquad (4.22)$$

and the KMS state $\varphi_{k/n}$ on O_C is given by $\varphi_{k/n} = \tau \circ \Phi$, where τ is the unique normalized trace on $\mathcal{F}_0$, which satisfies

$$\tau(S_w S_{w'}^*) = \delta_{w,w'} q^{-rk},$$

for w and w' words of length r. This agrees with the values of the KMS state we saw in (4.14) for $\beta = k/n$ and all the $\lambda_i = n$. Consider then the GNS representation π_φ associated to the KMS state φ on O_C. We denote by $\mathcal{M}$ the von Neumann algebra

$$\mathcal{M} = \pi_\varphi(O_C)''. \tag{4.23}$$

By rescaling the time evolution (4.15), the state φ becomes a KMS state at inverse temperature $\beta = 1$ for the time evolution

$$\alpha_t(S_a) = q^{itk} S_a. \tag{4.24}$$

In fact, we have

$$\varphi(S_a S_a^*) = q^{-k} = \varphi(\alpha_i(S_a^*) S_a).$$

Thus, up to inner automorphisms, α_t is the modular automorphism group for the von Neumann algebra $\mathcal{M}$, which shows that the algebra $\mathcal{M}$ is of type $\mathrm{III}_{q^{-k}}$. The fixed point subalgebra $\mathcal{M}_0$ for the time evolution α_t is the weak closure of $\mathcal{F}_0$. This gives a copy of the hyperfinite type II_1 factor $\mathcal{M}_0$ inside $\mathcal{M}$, with the restriction to $\mathcal{M}_0$ of the KMS state φ giving the von Neumann trace τ.

We then consider the projection $P_\pi = \sum_{a \in C \cap \pi} S_a S_a^*$ as an element in $\mathcal{M}_0$. We have seen that the value of the KMS state φ on P_π is

$$\varphi(P_\pi) = \tau(P_\pi) = q^{-k} \cdot \#(C \cap \pi) = q^{-k + \ell \dim_H(S_C \cap \pi)} = q^{-k + n \dim_H(S_C \cap S_\pi)},$$

which gives (4.20a) and (4.20b).

4.7. KMS states and phase transitions for a single code. As above, let $C \subset A^n$ be an $[n, k, d]_q$ code and let TO_C and O_C be the associated Toeplitz and Cuntz algebras, respectively with generators T_a and S_a, for $a \in C$, satisfying $T_a^* T_a = 1$ for TO_C, and $S_a^* S_a = 1$ and $\sum_a S_a S_a^* = 1$ for O_C.

In addition to the representations of O_C on $L^2(S_C, \mu_R)$ constructed previously, it is natural also to consider the Fock space representation of TO_C on the Hilbert space $\mathcal{H}_C = \ell^2(\mathcal{W}_C)$, where $\mathcal{W}_C$ is the set of all words of finite length in the elements $a \in C$,

$$\mathcal{W}_C = \cup_{m \geq 0} \mathcal{W}_{C,m},$$

with

$$\mathcal{W}_{C,m} = \{ w = w_1 \cdots w_m \,|\, w_i \in C \subset A^n \}$$

and $\mathcal{W}_{C,0} := \{\emptyset\}$. For all w, we identify the words $w\emptyset = w$. We denote by ϵ_w, for $w \in \mathcal{W}_C$, the canonical orthonormal basis of $\ell^2(\mathcal{W}_C)$. We also denote $\epsilon_\emptyset = \epsilon_0$.

4.7.1. Lemma. *The operators on* $\mathcal{H}_C$ *given by*

$$T_a \epsilon_w = \epsilon_{aw} \tag{4.25}$$

define a representation of the Toeplitz algebra TO_C *on* $\mathcal{H}_C$*.*

Proof. The adjoint T_a^* of the operator (4.25) is given by

$$T_a^* \epsilon_w = \delta_{a,w_1} \epsilon_{\sigma(w)}, \tag{4.26}$$

where δ_{a,w_1} is the Kronecker delta, and $\sigma(w) = w_2 \cdots w_m \in \mathcal{W}_{C,m-1}$, for $w = w_1 \cdots w_m \in \mathcal{W}_{C,m}$. In fact, we have

$$\langle T_a f, h \rangle = \sum_w \overline{f_{aw}} h_w = \sum_{w'=aw} \overline{f_{w'}} h_{\sigma(w')} = \sum_{w'} \overline{f_{w'}} \delta_{a,w'_1} h_{\sigma(w')} = \langle f, T_a^* h \rangle,$$

for $f = \sum_w f_w \epsilon_w$ and $h = \sum_w h_w \epsilon_w$ in $\mathcal{H}_C$. Thus, $T_a T_a^* = P_a$, where P_a is the projection onto the subspace $\mathcal{H}_{C,a}$ spanned by the ϵ_w with $w_1 = a$. One also has

$$T_a^* T_a f = \sum_w f_w T_a^* \epsilon_{aw} = f,$$

so that we obtain $T_a^* T_a = 1$.

This completes the proof.

We consider then time evolutions on the algebra TO_C associated to the random walks and Ruelle transfer operators introduced in §4.3 and 4.4.

4.7.2. Lemma. *Let* $W_\beta(x) = \exp(-\beta \lambda_{x_1})$*, for* $x \in S_C$*, be a potential satisfying the Keane condition* $\sum_{a\in C} \exp(-\beta\lambda_a) = 1$*. Then setting*

$$\sigma_t(T_a) = e^{it\lambda_a} T_a \tag{4.27}$$

defines a time evolution on the algebra TO_C*, which is implemented, in the Fock representation, by the Hamiltonian*

$$H\epsilon_w = (\lambda_{w_1} + \cdots + \lambda_{w_m})\, \epsilon_w, \qquad \text{for } w = w_1 \cdots w_m \in \mathcal{W}_{C,m}. \tag{4.28}$$

Proof. It is clear that (4.27) determines a 1–parameter group of continuous automorphisms of the algebra TO_C. The Hamiltonian that implements the time evolution in the Fock representation is a self adjoint unbounded operator on the Hilbert space $\mathcal{H}_C$ with the property that $\sigma_t(A) = e^{itH} A e^{-itH}$, for all elements $A \in TO_C$. We see on the generators T_a that

$$e^{itH} T_a e^{-itH} \epsilon_w = e^{it(\lambda_a + \lambda_{w_1} + \cdots + \lambda_{w_n})} e^{-it(\lambda_{w_1} + \cdots + \lambda_{w_n})} \epsilon_{aw}$$

implies that e^{itH}, with H as in (4.28) is the one-parameter group that implements the time evolution (4.27) in the Fock representation.

The proof is completed.

We consider in particular the time evolution associated to the uniform Hausdorff measure on the fractal S_C of dimension $R = k/n$.

4.7.3. Proposition. *For an $[n,k,d]_q$- code C, we consider the time evolution*

$$\sigma_t(T_a) = q^{itn} T_a$$

on the algebra TO_C. Then for all $\beta > 0$ there is a unique KMS$_\beta$ state on the resulting quantum statistical mechanical system.

(1) At low temperature $\beta > R$, this is a type I_∞ state, with the partition function given by $Z_C(\beta) = \mathrm{Tr}(e^{-\beta H}) = (1 - q^{(R-\beta)n})^{-1}$ and the Gibbs equilibrium state of the form

$$\varphi_\beta(A) = Z_C(\beta)^{-1}\, \mathrm{Tr}(A e^{-\beta H}). \tag{4.29}$$

(2) At the critical temperature $\beta = R$, the unique KMS$_\beta$ state is a type $III_{q^{-k}}$ factor state, which induces the unique KMS state on the Cuntz algebra O_C, and is determined by the normalized R-dimensional Hausdorff measure μ_R on S_C. It is given by the residue

$$\varphi_R(A) = \mathrm{Res}_{\beta=R} \mathrm{Tr}(A e^{-\beta H}). \tag{4.30}$$

(3) At high temperature the unique KMS state is also of type III and determined by the values $\varphi_\beta(T_w T_w^*) = e^{-\beta(\lambda_{w_1} + \cdots + \lambda_{w_m})}$, where $\lambda_a = n \log q$ for all $a \in C$.

(4) Only at the critical temperature $\beta = R$ the KMS state φ_R induces a KMS state on the quotient algebra O_C.

Proof. First notice that any KMS state at inverse temperature β must have the same values on elements of the form $T_w T_{w'}^*$. This can be seen from the KMS condition, inductively from

$$\varphi_\beta(T_a T_a^*) = \varphi_\beta(\sigma_{iR}(T_a^*) T_a) = q^{-Rn} \varphi_\beta(T_a^* T_a) = q^{-\beta n}.$$

This determines the state uniquely. So we see that at all $\beta > 0$ where the set of KMS states is non-empty it consists of a single element.

The Hamiltonian has eigenvalues $mn \log q$, for $m \in \mathbf{N}$, each with multiplicity $q^{mk_r} = \#\mathcal{W}_{C,m}$. Thus, the partition function of the time evolution is given by

$$Z_C(\beta) = \mathrm{Tr}(e^{-\beta H}) =$$

$$= \sum_m \sum_{w \in \mathcal{W}_{C,m}} \exp(-\beta(\lambda_{w_1} + \cdots + \lambda_{w_m})) =$$

$$= \sum_m q^{mk} q^{-\beta nm} = \sum_m q^{(R-\beta)nm}. \tag{4.31}$$

The series converges for inverse temperature $\beta > R$, with sum

$$Z_C(\beta) = (1 - q^{(R-\beta)n})^{-1}.$$

Thus, in the low temperature range $\beta > R$, one has an equilibrium state of the Gibbs form (4.29).

At the critical temperature $\beta = R$, we have a KMS$_\beta$ state of type III$_{q^{-k}}$, which is the unique KMS state on the algebra O_C

$$\varphi_R(A) = \int_{S_C} \Phi(A)\, d\mu_R, \tag{4.32}$$

which induces a KMS state on TO_C by pre–composing the expectation $\Phi : O_C \to \mathcal{A}(\hat{S}_C)$ with the quotient map $TO_C \to O_C$. Here we use again the identification of $\mathcal{A}(\hat{S}_C)$ with the maximal abelian subalgebra of O_C, and μ_R is the normalized R-dimensional Hausdorff measure on S_C. This means that the state φ_R has values

$$\varphi_R(T_w T_{w'}^*) = \delta_{w,w'} \mu_R(S_C(w)) = q^{-Rnm} = q^{-km},$$

for $w = w_1 \dots w_m$. To see that, at this critical temperature, the state is given by a residue (and can therefore be expressed in terms of Dixmier trace), it suffices to observe that the partition function $Z(\beta)$ has a simple pole at $\beta = R$ with residue $\mathrm{Res}_{\beta=R} Z(\beta) = 1$, so that we have

$$\mathrm{Res}_{\beta=R} \mathrm{Tr}(T_w T_{w'}^*\, e^{-\beta H}) = e^{-\beta(\lambda_{w_1} + \cdots + \lambda_{w_m})} \mathrm{Res}_{\beta=R} Z(\beta) = \varphi_R(T_w T_{w'}^*).$$

At higher temperatures $\beta < R$ the KMS state is similarly determined by the list of values

$$\varphi_R(T_w T_{w'}^*) = \delta_{w,w'} e^{-\beta(\lambda_{w_1}+\cdots+\lambda_{w_m})} = \delta_{w,w'} q^{-\beta nm}.$$

To see that only the state at critical temperature induces a KMS state on the quotient algebra O_C it suffices to notice that in O_C one has the additional relation $\sum_a S_a S_a^* = 1$, which requires that the values of a KMS$_\beta$ state satisfy the Keane relation

$$\sum_a \varphi_\beta(S_a S_a^*) = \sum_a e^{-\beta\lambda_a} = 1.$$

This is satisfies at $\beta = R$, where it gives the self-similarity relation for the Hausdorff dimension of the fractal S_C, but it is not satisfied at any other $\beta \neq R$.

The proof is complete.

We see from the above result that the situation is very similar to the one encountered in the construction of the Bost–Connes system [BoCo], where the case of the system without interaction is obtained as a tensor product of Toeplitz algebras (in that case in a single generator) with their unique KMS$_\beta$ state at each $\beta > 0$. We explain below how a similar approach with tensor products plays a role here in describing the curve $R = \alpha_q(\delta)$ in terms of phase transitions.

4.8. Crossed product description. Before we discuss families of codes and tensor products of quantum statistical mechanical systems, it is worth reformulating the setting described above in a way that may make it easier to pass to the analog of the "systems with interaction" of [BoCo].

Let C be an $[n,k,d]_q$ code. We introduce the notation $\Xi_C(P)$ for the algebra obtained by compressing the abelian subalgebra $T\mathcal{A}(C) \subset TO_C$ with a projection P of TO_C,

$$\Xi_C(P) := P\; T\mathcal{A}(C)\; P.$$

The isometries T_a, for $a \in C$, determine an endomorphism ρ of the algebra $T\mathcal{A}(C)$ given by

$$\rho(X) = \sum_a T_a\, X\, T_a^*. \tag{4.33}$$

This endomorphism satisfies $\rho(1) = P$, the idempotent $\sum_a T_a T_a^* = P$ in $T\mathcal{A}(C) \subset TO_C$. The endomorphism ρ has partial inverses σ_a given by

$$\sigma_a(X) = T_a^* X T_a, \tag{4.34}$$

for $X \in \Xi_C(P_a)$, where $P_a = T_a T_a^*$ is the range projection. They satisfy

$$\sigma_a \rho(X) = X, \quad \forall X \in T\mathcal{A}(C). \tag{4.35}$$

Notice that, for $X = T_w T_w^*$ in $T\mathcal{A}(C)$, we have $PX = T_{w_1} T_{w_1}^* T_w T_w^* = T_w T_w^* = X$ and $XP = T_w T_w^* T_{w_1} T_{w_1}^* = T_w T_w^* = X$, so that, if one represents an arbitrary element $X \in T\mathcal{A}(C)$ in the form $X = \lambda_0 + \sum_w \lambda_w T_w T_w^*$, one finds $PX = XP = \sum_a \lambda_0 T_a T_a^* + \sum_w \lambda_w T_w T_w^*$. Similarly, one has $\rho(X) = \lambda_0 P + \sum_{aw} \lambda_w T_a T_w T_w^* T_a^*$, which acts as a shift on the coefficients λ_w and lands in the compressed algebra $\Xi_C(P)$. The partial inverses σ_a satisfy $\sigma_a(1) = 1$ since $T_a^* T_a = 1$, and they map an element $X = \lambda_0 + \sum_w \lambda_w T_w T_w^*$ of $T\mathcal{A}(C)$ to $\sigma_a(X) = \lambda_0 + \sum_{w=aw'} \lambda_w T_{w'} T_{w'}^*$.

In the case of the quotient algebra O_C, where one imposes the relations $S_a^* S_a = 1$ and $\sum_a S_a S_a^* = 1$, the endomorphism above induces an endomorphism $\bar{\rho}$ of the algebra $\mathcal{A}(\hat{S}_C)$ with $\bar{\rho}(1) = 1$, which is given simply by the composition

$$\bar{\rho}(f) = \sum_a S_a \, f \, S_a^* = f \circ \sigma$$

with the one-sided shift map $\sigma : S_C \to S_C$,

$$\sigma(x_1 x_2 \cdots x_m \cdots) = x_2 x_3 \cdots x_{m+1} \cdots$$

and the partial inverses are the compositions with the partial inverses of the one sides shift

$$\bar{\sigma}_a(f) = S_a^* \, f \, S_a = f \circ \sigma_a,$$

where $\sigma_a(x_1 x_2 \cdots x_m \cdots) = a x_1 x_2 \cdots x_m \cdots$.

Thus, we can form the crossed product algebra $T\mathcal{A}(C) \rtimes_\rho \mathbf{M}$, where $\mathbf{M}$ is the additive monoid $\mathbf{M} = \mathbf{Z}^+$. This has generators $T_w T_w^*$ together with an extra generator S satisfying $S^* S = 1$ and $S X S^* = \rho(X)$. It also satisfies $SS^* = P$ and $S^* X S = \sigma_a(X)$, for $X \in \Xi_C(P_a)$.

4.8.1. Proposition. *The morphism $\Psi : TO_C \to T\mathcal{A}(C) \rtimes_\rho \mathbf{M}$ defined by setting*

$$\Psi(T_a) = P_a S \tag{4.36}$$

identifies TO_C with the subalgebra $\Xi_C(P) \rtimes_\rho \mathbf{M}$. On the quotient algebra O_C, the induced morphism $\bar{\Psi}$ gives an isomorphism $O_C \simeq \mathcal{A}(\hat{S}_C) \rtimes_\rho \mathbf{M}$.

Proof. Notice that $\Psi(T_a)^*\Psi(T_a) = S^* P_a S = \sigma_a(P_a) = T_a^* P_a T_a = 1$ and $\sum_a \Psi(T_a)\Psi(T_a)^* = \sum_a P_a S S^* P_a = \sum_a P_a P P_a = \sum_a P_a = P$, since, as observed above, $P_a P = P P_a = P_a$. Thus, Ψ maps injectively $TO_C \subset T\mathcal{A}(C) \rtimes_\rho \mathbf{M}$. To see that surjectivity also holds, notice that $\Xi_C(P) \rtimes_\rho \mathbf{M}$ is spanned linearly by monomials of the form $T_w T_w^* S^k$ and $S^k T_w T_w^*$, for $w \in \mathcal{W}_{C,m}$, $m \geq 1$, and $k \geq 0$. It suffices to show that these are all in the range of the map Ψ. First observe that the map Ψ is the identity on the subalgebra $T\mathcal{A}(C) \subset TO_C$. In fact, for $w = w_1 \cdots w_m$, with $w_i \in C$, we have

$$\Psi(T_w T_w^*) = P_{w_1}\rho(P_{w_2})\cdots\rho^{m-1}(P_{w_m})(SS^*)^m \rho^{m-1}(P_{w_m})\cdots P_{w_1}$$

$$= P_w P P_w = P_w = S_w S_w^*.$$

Notice then that we have $\Psi(\sum_a T_a) = \sum_a P_a S = PS$. Let $Y = \sum_a T_a$ in TO_C. We then have

$$\Psi(T_w T_w^*)\Psi(Y^k) = T_w T_w^* (PS)^k.$$

We have $(PS)^k = P \ldots \rho^{k-1}(P) S^k$. Since $P = SS^*$ and $\rho(X) = SXS^*$, we see that $P\rho(P) = \rho(P)$ and $P\rho(P)\cdots\rho^{k-1}(P) = \rho^{k-1}(P) = S^{k-1}S^{*k-1}$. Thus, $\rho^{k-1}(P)S^k = S^k$ and we obtain that

$$\Psi(T_w T_w^* Y^k) = T_w T_w^* S^k.$$

The argument for elements of the form $S^k T_w T_w^*$ is analogous. Thus, all the monomials with $w \in \mathcal{W}_{C,m}$ with $m \geq 1$ are in the range of Ψ and the only missing terms are the S^k and their adjoints (the case of $w = \emptyset \in \mathcal{W}_{C,0}$).

This induces the isomorphism $O_C \simeq \mathcal{A}(\hat{S}_C) \rtimes_{\bar\rho} \mathbf{M}$ of [Exel], where in the quotient algebra $\bar{S}^* f \bar{S} = q^{-k}\sum_a f \circ \sigma_a$ is the Perron–Frobenius operator and the induced map $\bar\Psi$ preserves the additional relation $\sum_a S_a S_a^* = 1$. Thus, in this case we have $\bar\Psi(\sum_a S_a) = \sum_a P_a \bar{S} = \bar{S}$, since in this case $\bar{P} = \sum_a S_a S_a^* = 1$. We then obtain that the range of $\bar\Psi$ is all of $\mathcal{A}(\hat{S}_C) \rtimes_{\bar\rho} \mathbf{M}$. This completes the proof.

With this description of the algebra TO_C in terms of crossed product of $\Xi_C(P)$ by the monoid $\mathbf{M}$, one can view the time evolution as given by

$$\sigma_t(X) = X, \quad \text{for} \quad X \in \Xi_C(P), \quad \text{and} \quad \sigma_t(S) = q^{itn} S. \tag{4.37}$$

5. Quantum statistical mechanics and Kolmogorov complexity

Our reformulation of the rate and relative minimum distance of codes in terms of Hausdorff dimensions, as well as the construction of algebras with time evolutions for individual codes, can be reinterpreted within the context of Kolmogorov complexity and Levin's universal enumerable semi-measures.

5.1. Languages and fractals. We begin with some considerations on structure functions and entropies for codes. Suppose given a code $C \subset A^n$, for an alphabet A with $\#A = q$. We assume that C is an $[n,k,d]_q$ code.

First we reinterpret the construction of the fractal S_C in terms of languages and ω-languages.

Given the alphabet A, one writes $A^\infty = \cup_n A^n$ for the set of all words of finite length in the alphabet A and one denotes by A^ω the set of all words of infinite length in the same alphabet. A language Λ is a subset of A^∞ and an ω-language is a subset of A^ω.

To a code C one can associate a language Λ_C given by all words in A^∞ that are successions of words in $C \subset A^n$. Similarly, one has an ω-language Λ_C^ω given by all infinite words in A^ω that are a succession of elements in C. As such, the ω-language Λ_C^ω is set-theoretically identified with the fractal $\hat{S}_C$ we considered previously.

There is a notion of entropy for languages ([Eilen], see also the recent [Sta3]), which is defined as follows. One first introduces the structure function

$$s_\Lambda(m) = \#\{w \in \Lambda \,:\, \ell(w) = m\},$$

the number of words of length m in the language Λ. These can be assembled together into a generating function

$$G_\Lambda(t) = \sum_m s_\Lambda(m) t^m.$$

The entropy of the language Λ is then the log of the radius of convergence of the series above

$$\mathcal{S}_\Lambda = -\log_{\#A} \rho(G_\Lambda).$$

5.1.1. Lemma. *For the language Λ_C defined by an $[n,k,d]_q$-code C the structure function satisfies*

$$G_{\Lambda_C}(q^{-\beta}) = Z_C(\beta),$$

where $Z_C(\beta)$ is the partition function of the quantum statistical mechanical system (TO_C, σ_t) associated to the code C. The entropy of the language Λ_C is the rate of the code $\mathcal{S}_{\Lambda_C} = k/n = R$.

Proof. In the case of an $[n,k,d]_q$-code C, notice that the series G_{Λ_C} is given by

$$G_{\Lambda_C}(t) = \sum_m q^{km} t^{nm} = (1 - q^k t^n)^{-1},$$

since one has $s_\Lambda(N) = 0$ for $N \neq mn$, while for $N = mn$ one has $s_\Lambda(nm) = q^{km}$. In particular, when expressed in the variable $t = q^{-s}$ this becomes

$$G_{\Lambda_C}(q^{-s}) = \sum_m q^{(R-s)nm} = (1 - q^{(R-s)n})^{-1},$$

with convergence for $\beta = \Re(s) > R$. This recovers the partition function $Z_C(\beta)$ of the quantum statistical mechanical system associated to the code C. This gives an entropy

$$\mathcal{S}_{\Lambda_C} = -\log_q \rho(G_\Lambda) = R = k/n,$$

since domain of convergence for $\beta > R$ corresponds to $|t| = |q^{-s}| < q^{-R}$.

Intersection with linear spaces π_ℓ determines induced languages $\Lambda_{C,\ell}$. The threshold value $\ell = d$ corresponds to the minimal dimension for there is a choice of π_d for which the resulting language is non-trivial, with entropy d.

5.2. Kolmogorov complexity. There are several variants of Kolmogorov complexity for words w of finite length in a given alphabet, see [LiVi], §5.5.4. To any such complexity function $K(w)$ one associates the lower Kolmogorov complexity for infinite words by setting

$$\kappa(x) = \liminf_{w \to x} \frac{K(w)}{\ell(w)},$$

where the limit is taken over finite words w that are truncations of increasing length $\ell(w) = m \to \infty$ of an infinite word x. There is a characterization (see [ZvoLe] and [LiVi]) of the lower Kolmogorov complexity in terms of measures, which we discuss more at length in the case of codes here below.

We begin by recalling the notion of semi-measures and provide examples taken from the constructions we have already seen in the previous sections of this paper.

5.2.1. Definition. *A semi-measure on S_C is a positive real valued function on the cylinder sets $\{S_C(w)\}$ that satisfies $\mu(S_C) \leq 1$ and the subadditivity property*

$$\mu(S_C(w)) \geq \sum_{a \in C} \mu(S_C(wa)).$$

Here we do not distinguish between $\hat{S}_C = \Lambda_C^\omega$ and S_C since the difference is of measure zero in any of the above measures. An example of semi-measures is obtained using the Ruelle transfer operator techniques considered above.

5.2.2. Lemma. *Let $W_\beta(x)$ be a potential that satisfies the Keane condition at $\beta = \beta_0$ and such that, for a fixed x, it is monotonically decreasing as a function of β. Then the function*

$$\mu_{x_0,\beta}(S_C(w)) = W_\beta(w_1 x_0) \cdots W_\beta(w_n \cdots w_1 x_0)$$

is a semi-measure.

Proof. Suppose given a potential $W_\beta(x)$, and assume that for a $\beta = \beta_0$ it satisfies the Keane condition $\sum_{a\in C} W_{\beta_0}(ax) = 1$. Assume, moreover, that for fixed $x \in S_C$, the function $W_\beta(x)$ is monotonically decreasing as a function of β. This will certainly be the case for the special cases we considered with $W_\beta(x) = e^{-\beta\lambda_{x_1}}$ of $W_\beta(x) = e^{-\beta\lambda_{x_1 x_2}}$. One will then have

$$\sum_{a\in C} W_\beta(ax) \leq 1, \quad \text{ for } \quad \beta \geq \beta_0, \quad \forall x \in S_C.$$

Thus, one has

$$\sum_{a\in C} \mu(S_C(wa)) = \sum_{a\in C} W_\beta(w_1 x_0) \cdots W_\beta(w_n \cdots w_1 x_0) \cdot W_\beta(a w_n \cdots w_1 x_0)$$

$$\leq W_\beta(w_1 x_0) \cdots W_\beta(w_n \cdots w_1 x_0) = \mu_{x_0,\beta}(S_C(w)),$$

for all $\beta \geq \beta_0$. This completes the proof.

5.3. Enumerable semi-measures. In complexity theory one is especially interested in those semi-measures that are enumerable. We recall here a characterization of enumerable semi-measure given in Theorem 4.5.2 of [LiVi], which will be useful in the following,

Given a language Λ, let $\mathcal{F}_\Lambda$ be the class of functions (called monotone in [LiVi]) $f : A^\infty \to \Lambda$, where A^∞ is the set of all finite words (of arbitrary length) in the alphabet A, with $f(ww') = f(w)f(w')$, the product being concatenation of words in Λ. These extend to functions from A^ω, the set of all infinite words in the alphabet A to the ω-language Λ^ω.

Given a semi-measure μ on A^ω and a function $f \in \mathcal{F}_\Lambda$ one obtains a semi-measure μ_f on Λ^ω by setting

$$\mu_f(\Lambda^\omega(w)) = \sum_{w' \in A^\infty : f(w') = w} \mu(A^\omega(w')),$$

where, as usual, $\Lambda^\omega(w)$ and $A^\omega(w')$ denote the subsets of Λ^ω and A^ω, respectively, made of infinite words starting with the given prefix word w or, respectively, w'.

In particular, let λ denote the 1-dimensional Lebesgue measure on $[0,1]$. This induces a measure on A^ω by mapping the infinite sequences in A^ω to points of $[0,1]$ written in their q-ary expansion. The measure satisfies

$$\lambda(A^\omega(w)) = q^{-\ell(w)},$$

where $\ell(w)$ is the length of the word $w \in A^\infty$.

Then Theorem 4.5.2 of [LiVi] characterizes enumerable semi-measures on Λ^ω as those semi-measures μ that are obtained as $\mu = \lambda_f$ for a function $f \in \mathcal{F}_\Lambda$.

We observe first that these measures satisfy the following multiplicative property. For simplicity of notation, we write in the following $\mu(w)$ for $\mu(\Lambda^\omega(w))$.

5.3.1. Lemma. *The enumerable semi-measures are multiplicative on concatenations of words, $\mu(ww') = \mu(w)\mu(w')$.*

Proof. The uniform Lebesgue measure λ clearly has that property since $\lambda(ww') = q^{-\ell(ww')} = q^{-(\ell(w)+\ell(w'))} = \lambda(w)\lambda(w')$. Suppose then given a function $f \in \mathcal{F}_\Lambda$. This satisfies $f(ww') = f(w)f(w')$ by definition. Thus, in particular, we can write $f(w) = f(w_1)\cdots f(w_m)$, for a word $w = w_1 \cdots w_m$ of length $\ell(w) = m$. Consider then the measure $\mu = \lambda_f$ given by $\lambda_f(u) = \sum_{w: f(w)=u} \lambda(w)$. For a word $u = u_1 \cdots u_m$ of length $\ell(u) = m$, we can then write this equivalently as

$$\lambda_f(u) = \sum_{f(w_i)=u_i} \prod_i \lambda(w_i) = \prod_{i=1}^m \lambda_f(u_i).$$

This completes the proof.

The characterization of enumerative semi-measure as semi-measures of the form $\mu = \lambda_f$ shows, for example, that the uniform Hausdorff measure of dimension $\dim_H S_C = R = k/n$ on the set S_C considered above is an enumerative (semi)-measure. In fact, it is of the form $\mu = \lambda_f$, where the map f is induced by the coding map $E : A^k \to C \subset A^n$, so that elements $a \in C$ are described as $a = f(w)$ for a word $w \in A^k$. In this case, since the coding map E is injective, there is a unique word w with $f(w) = a$.

Another example of an enumerative (semi)-measure on S_C can be obtained using as function $f \in \mathcal{F}_\Lambda$ the decoding map P, by which we mean the map that assigns to each element in A^n the nearest point in C in the Hamming metric. Then one obtains

$$\mu_f(S_C(u)) = \sum_{w=(w_i):w_i\in A^n\,,\,P(w_i)=u_i} \lambda(w) = \#\{w = (w_i) : P(w_i) = u_i\} q^{-nm},$$

for $u = u_1 \cdots u_m$ with $u_i \in C$, and $w = w_1 \cdots w_m$ with $w_i \in A^n$.

We now connect enumerable semi-measures on S_C to quantum statistical mechanical systems on the Toeplitz–Cuntz algebra TO_C in the following way.

5.3.2. Lemma. *Let μ be a semi-measure on S_C such that $\mu(ww') = \mu(w)\mu(w')$, where $\mu(w)$ is shorthand for $\mu(S_C(w))$. Then setting*

$$\sigma_t(T_a) = \mu(S_C(a))^{-it} T_a$$

determines a time evolution $\sigma_t \in Aut(TO_C)$. In the Fock space representation of TO_C, this time evolution has Hamiltonian

$$H\epsilon_w = -\log \mu(S_C(w))\, \epsilon_w.$$

The partition function is

$$Z_{\mu,C}(\beta) = (1 - \sum_{a\in C} \mu(S_C(a))^\beta)^{-1},$$

with a pole at a critical $\beta_c \le 1$, the inverse temperature at which $\sum_a \mu(a)^{\beta_c} = 1$. The functional

$$\varphi(T_w T_{w'}^*) = \delta_{w,w'}\, \mu(S_C(w))^\beta$$

is a KMS_β state for the quantum statistical mechanical system (TO_C, σ).

Proof. In the Fock representation the time evolution is generated by a Hamiltonian

$$e^{itH} T_a e^{-itH} \epsilon_w = \sigma_t(T_a)\epsilon_w = \mu(a)^{-it} \epsilon_{aw},$$

which gives

$$e^{itH}\epsilon_w = \mu(w)^{-it}\epsilon_w$$

using the fact that the semi-measure satisfies $\mu(aw) = \mu(a)\mu(w)$. This gives $H\epsilon_w = -\log \mu(w)$. The partition function is then given by

$$Z_{\mu,C}(\beta) = Tr(e^{-\beta H}) = \sum_w \mu(w)^\beta.$$

Again using $\mu(w) = \mu(w_1)\cdots\mu(w_m)$ for $w = w_1 \cdots w_m$ a word of length $\ell(w) = m$, we write the above as

$$\sum_w \mu(w)^\beta = \sum_m \sum_{w_1,\dots,w_m} \mu(w_1)^\beta \cdots \mu(w_m)^\beta = \sum_m (\sum_{a\in C} \mu(a)^\beta)^m.$$

For $\beta > \beta_c$ where $\sum_a \mu(a)^{\beta_c} = 1$, the series converges to

$$Z_{\mu,C}(\beta) = (1 - \sum_{a\in C} \mu(a)^\beta)^{-1}.$$

Since μ is a semi-measure, it satisfies $\sum_a \mu(a) \leq 1$, so that $\beta_c \leq 1$. The state defined by the condition $\varphi(T_w T_{w'}^*) = \delta_{w,w'}\mu(w)^\beta$ satisfies the KMS_β condition. This can be checked inductively from

$$\varphi(T_a T_a^*) = \mu(a)^\beta = \mu(a)^\beta \varphi(T_a^* T_a) = \varphi(T_a^* \sigma_{i\beta}(T_a)).$$

This completes the proof.

This result in particular shows that, given a semi-measure on S_C with strict inequality $\sum_a \mu(a) < 1$, there is a way to associate to it a measure by raising the temperature, that is, lowering β from $\beta = 1$ to $\beta = \beta_c$. One then has $\varphi(S_w S_w^*) = \mu(w)^{\beta_c}$, this time satisfying the correct normalization $\sum_a \mu(a)^{\beta_c} = 1$, which also implies

$$\sum_a \mu(wa)^{\beta_c} = \mu(w)^{\beta_c} \sum_a \mu(a)^{\beta_c} = \mu(w)^{\beta_c},$$

so that one indeed obtains a measure.

5.4. Universal enumerable semi-measure. A well known result of Levin (see [ZvoLe] or Theorem 4.5.1 of [LiVi]) is that there exist universal (or maximal) enumerable semi-measures μ_U on Λ^ω. They are characterized by the following property: any enumerable semi-measure μ is absolutely continuous with respect to μ_U with bounded Radon-Nikodym derivative, or equivalently $\mu_U \geq c_f \lambda_f$, for all $f \in \mathcal{F}_\Lambda$. Such universal semi-measures are not unique. A way to construct one is by listing the enumerable semi-measures (or equivalently listing the functions $f \in \mathcal{F}_\Lambda$) and then taking $\mu_U = \sum_n \alpha_n \lambda_{f_n}$ with positive real coefficients α_n with $\sum_n \alpha_n \leq 1$, see Theorem 4.5.1 of [LiVi]. Another description which is more suitable for our purposes is as an enumerable semi-measure $\mu_U = \lambda_{f_U}$, where f_U is a *universal monotone machine* in the sense of Definitions 4.5.2 and 4.5.6 of [LiVi], that is, universal for Turing machines with a one-way read-only input tape, some work tapes, and a one-way write-only output tape. As an enumerable semi-measure, we can apply to it the construction of a corresponding time evolution and quantum statistical mechanical system as above. Notice that μ_U is not recursive and it is not a measure, that is, the inequality $\sum_a \mu_U(a) < 1$ is strict, see Lemma 4.5.3 of [LiVi].

We can then consider on the Toeplitz–Cuntz algebra TO_C the *universal time evolution*

$$\sigma_t(T_a) = \mu_U(a)^{-it} T_a$$

induced by the universal enumerable semi-measure $\mu_U = \lambda_{f_U}$. The critical value $\beta_U < 1$ at which the partition function

$$Z_{U,C}(\beta) = (1 - \sum_a \mu_U(a)^\beta)^{-1}$$

has a pole is the *universal critical inverse temperature*. This universal critical temperature can be regarded as another parameter of a code C, which in this setting replaces the code rate R as the critical β is the time evolution.

The universal critical inverse temperature β_U can also be described as a Hausdorff dimension, by modifying the construction of the Sierpinski fractal S_C associated to the code C in the following way.

Recall that S_C is constructed inductively starting with the space $(0,1)^n_q$ viewed as $(\infty \times n)$-matrices with entries in A. At the first step, replacing it by q^k copies scaled down by a factor of q^{-n}, each identifies with the subset $(0,1)^n_{q,a}$ of points in $(0,1)^n_q$ where the first row is equal to the element $a \in C$, with $C \subset A^n$. Each $(0,1)^n_{q,a}$ is a copy of $(0,1)^n_q$ scaled down by a factor of q^{-n}. One obtains then S_C by iterating this process on each $(0,1)^n_{q,a}$ and so on.

Now we consider a very similar procedure, where we again start with the same set $(0,1)^n_q$. We again consider all the subsets $(0,1)^n_{q,a}$ as above, but where the set

$(0,1)^n_{q,a}$ is metrically a scaled down copy of $(0,1)^n_q$, now scaled by a factor $\mu_U(a)$ instead of being scaled by the uniform factor q^{-n} as in the construction of S_C. One obtains in this way a fractal $S_{C,U}$, by iterating this process. The self similarity equation for the non-uniform fractal $S_{C,U}$ is then given by

$$\sum_{a\in C} \mu_U(a)^s = 1,$$

which identifies its Haudorff dimension with $s = \beta_U$.

One also has a Ruelle transfer operator associated to the universal enumerable semi-measure, which is given by

$$\mathcal{R}_{\sigma,U,\beta} f(x) = \sum_{a\in C} \mu_U(a)^\beta f(ax).$$

It is then natural to investigate how the universal enumerative semi-measure is related to the Hausdorff dimension $\dim_H S_C = R$ and to Kolmogorov complexity.

5.4.1. Lemma. *For all words $x \in \hat{S}_C$ the lower Kolmogorov complexity is bounded above by*

$$\kappa(x) \leq \dim_H(S_C) = R.$$

Proof. The universal enumerable semi-measure μ_U is related to the lower Kolmogorov complexity by ([UShe], [ZvoLe], [Sta3])

$$\kappa(x) = \liminf_{w\to x} \frac{-\log_q \mu_U(w)}{\ell(w)},$$

where again the limit is taken over finite length truncations w of the infinite word x as the length $\ell(w)$ goes to infinity. We know by construction that the universal μ_U dominates multiplicatively all the enumerable semi-measures. Thus, in particular, if μ is the Hausdorff measure on S_C of dimension $R = \dim_H(S_C)$, which we have seen above is an enumerable (semi)-measure, there is a positive real number α such that $\mu_U(w) \geq \alpha\mu(w)$, for all finite words w. This implies that

$$\frac{-\log_q \mu_U(w)}{\ell(w)} \leq \frac{-\log_q \mu(w)}{\ell(w)} + \frac{-\log_q \alpha}{\ell(w)}.$$

This gives

$$\liminf_{w\to x} \frac{-\log_q \mu_U(w)}{\ell(w)} \leq \lim_{w\to x} \frac{-\log_q \mu(w)}{\ell(w)} = \lim_{m\to\infty} \frac{km}{nm} = R.$$

Moreover, we have the following result.

5.4.2. Lemma. *The lower Kolmogorov complexity satisfies*

$$\sup_{x\in\hat{S}_C} \kappa(x) = R$$

with the supremum achieved on a set of full measure.

Proof. This follows directly from Ryabko's inequality [Rya1], [Rya2], which shows that in general one has the estimate

$$\dim_H(\Lambda^\omega) \leq \sup_{x\in\Lambda^\omega} \kappa(x).$$

To see this more explicitly in our case, recall first that the Hausdorff dimension of a set X embedded in some larger ambient Euclidean space can be computed in the following way. Consider coverings $\{U_\alpha\}$ of X with diameters $diam(U_\alpha) \leq \rho$ and consider the sum $\sum_\alpha diam(U_\alpha)^s$. Set

$$\ell_s(X,\rho) = \inf\{\sum_\alpha diam(U_\alpha)^s \, : \, diam(U_\alpha) \leq \rho\}.$$

Then one has

$$\dim_H(X) = \inf\{s \, : \, \lim_{\rho\to 0} \ell_s(X,\rho) = 0\} = \sup\{s \, : \, \lim_{\rho\to 0} \ell_s(X,\rho) = \infty\}.$$

We then use an argument similar to the one used in [Rya2]: from

$$\kappa(x) = \liminf_{w\to x} \frac{-\log_q \mu_U(w)}{\ell(w)}$$

we know that, for a given $x \in S_C$, and for arbitrary $\delta > 0$, there is an integer $m(x)$ such that, if $w(x)$ denotes the truncation of length $m(x)$ of the infinite word x then

$$\frac{-\log_q \mu_U(w(x))}{m(x)} \leq \kappa(x) + \delta \leq \kappa + \delta,$$

where $\kappa = \sup_x \kappa(x)$ as above. The integer $m(x)$ can be taken so that $q^{-m(x)} \leq \rho$ for a given size $\rho \in \mathbf{R}_+^*$. Let $\mathcal{L}$ be the countable set of words $w = w(x)$ of lengths $m(x)$, for $x \in S_C$, obtained as above. We can then construct a covering of S_C with sets $S_C(w)$, for $w \in \mathcal{L}$, with diameters $diam(S_C(w)) = \sqrt{n}\, q^{-m(x)} \leq \sqrt{n}\, \rho$, for a positive constant α that only depends on n. These satisfy

$$\sum_{w \in \mathcal{L}} diam(S_C(w))^{\kappa+\delta} \leq \alpha \sum q^{-m(x)(\kappa+\delta)},$$

with $\alpha = \sqrt{n}^{(\kappa+\delta)}$. This gives

$$\sum q^{-m(x)(\kappa+\delta)} \leq \sum q^{m(x) \frac{\log_q \mu_U(w(x))}{m(x)}} \leq \sum_{w \in \mathcal{L}} \mu_U(w) \leq 1.$$

We then have

$$\ell_s(S_C, \rho) \leq \sum_{w \in \mathcal{L}} diam(S_C(w))^s$$

and therefore

$$\lim_{\rho \to 0} \ell_s(S_C, \rho) \leq \sum_{w \in \mathcal{L}} diam(S_C(w))^s.$$

For $s = \kappa+\delta$ the right hand side is uniformly bounded above, so $\lim_{\rho\to 0} \ell_{\kappa+\delta}(S_C, \rho) < \infty$, hence $\kappa+\delta \geq \dim_H(S_C)$, hence $\kappa \geq \dim_H(S_C)$, since δ can be chosen arbitrarily small.

6. Functional analytic constructions for limit points

6.1. Realizing limit points of the code domain. We have seen in the previous sections that, given an $[n,k,d]_q$ code C, one can construct fractal sets S_C and S_π as in §3.3, that have Hausdorff dimension, respectively, equal to $R = k/n$ and $\delta = d/n$, and that the parameter d can be characterized in terms of the behavior of the Hausdorff dimension of the intersections $S_{C,\ell,\pi} = S_C \cap S_\pi$ for π of dimension ℓ. We now consider the case where two assigned values R and δ are not necessarily realized by a code C, but are an accumulation point of the code domain, namely there exists an infinite family C_r of $[n_r, k_r, d_r]_q$ codes, where $k_r/n_r \to R$ and $d_r/n_r \to \delta$ as $r \to \infty$.

We show here that one can still construct sets S_R and S_δ, depending on the approximating family C_r, with the property that $\dim_H(S_R) = R$ and $\dim_H(S_\delta) = \delta$

and so that these sets are, in a suitable sense, approximated by the sets S_{C_r} and S_{π_r} with $\pi_r \in \Pi_{d_r}$ of the family of codes C_r.

6.2. Multifractals in infinite dimensional cubes. Let then $(0,1)_q^\infty$ denote the union $(0,1)_q^\infty = \cup_n (0,1)_q^n$ which can be considered as direct limit under the inclusion maps that embed $[0,1]^n \subset [0,1]^{n+1}$ as the face in $[0,1]^{n+1}$ of which the last coordinate is equal to zero. This is a metric space with the induced metric. In terms of the q-ary expansion, elements in $(0,1)_q^\infty$ can be described as infinite matrices with only finitely many columns with non zero entries. We can embed all the $S_{C_r} \subset (0,1)_q^{n_r}$ of an approximating family inside $(0,1)_q^\infty$. Thus, we can view the set $S_R = \cup_r S_{C_r}$ as $S_R \subset (0,1)_q^\infty$.

6.2.1. Proposition. *(1) For any limit point (R,δ) of the code domain there exists a family C_r of $[n_r,k_r,d_r]_q$ codes with $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$. (2) For such a sequence C_r the sets $S_R = \cup_r S_{C_r}$ and $S_\delta = \cup_r S_{\pi_{d_r}}$ have*

$$\dim_H(S_R) = R, \qquad \dim_H(S_\delta) = \delta, \qquad \textit{and} \qquad \dim_H(S_R \cap S_\delta) > 0. \tag{6.1}$$

(3) Moreover, given a sequence $\pi_{\ell_r} \in \Pi_{\ell_r}^{(n_r)}$ with $\ell_r \leq d_r - 1$, one can form the analogous $S_\ell = \cup_r S_{\pi_{\ell_r}}$. This has the property that $\dim_H(S_R \cap S_\ell) = 0$.

Proof. (1) We first show that we can find an approximating family C_r with $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$. To this purpose we use the spoiling operations on codes described above. We know from Corollary 1.2.1 that, given an $[n,k,d]_q$ code, we can produce an $[n, k-1 \leq k' \leq k, d-1]_q$ code from it by applying the second and third spoiling operations and twice the first one. Starting with an approximating family C_r with $k_r/n_r \to R$ and $d_r/n_r \to \delta$ and using the spoiling operations as described, we can produce from it other approximating families with k_r replaced by $k_r - \ell_r$ and $d_r - \ell_r$ with $\ell_r/n_r \to 0$ and such that, for sufficiently large r, $k_r/n_r - \ell_r/n_r \leq R$ and $d_r/n_r - \ell_r/n_r \leq \delta$. Possibly after passing to a subsequence, we obtain a family where the new k_r and d_r satisfy $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$.

(2) The Hausdorff dimension of a union behaves like

$$\dim_H(\cup_r X_r) = \sup_r \dim_H(X_r)$$

by countable stability ([Fal], p. 37). Thus, if $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$, we obtain that $\dim_H(S_R) = R$ and $\dim_H(S_\delta) = \delta$.

Let us now show that $\dim_H(S_R \cap S_\delta) > 0$. We have $S_R \cap S_\delta = \cup_r (S_{C_r} \cap S_{\pi_d})$. Again by countable stability of the Hausdorff dimension we obtain

$$\dim_H(S_R \cap S_\delta) = \sup_r \dim_H(S_{C_r} \cap S_{\pi_d}) > 0.$$

The Hausdorff dimension is also bounded above by the dimension of S_R and S_δ so $0 < \dim_H(S_R \cap S_\delta) \leq \min\{R, \delta\}$.

(3) For a given sequence $\ell_r \leq d_r - 1$ with corresponding linear spaces $\pi_{\ell_r} \in \Pi^{(n_r)}_{\ell_r}$, we can form the sets $S_{\pi_{\ell_r}} \subset (0,1)^\infty_q$. If the ℓ_r are chosen so that the ratio sequence $\ell_r/n_r \nearrow \ell$ approaches a limit from below as $r \to \infty$, then the same argument given above shows that the Hausdorff dimension $\dim_H(\cup_r S_{\pi_{\ell_r}}) = \ell$. For $S_\ell = \cup_r S_{\pi_{\ell_r}}$, the intersection $S_R \cap S_\ell$ is given as above by $S_R \cap S_\ell = \cup_r (S_{C_r} \cap S_{\pi_{\ell_r}})$. Since $\ell_r \leq d_r - 1$, we know that $\dim_H(S_{C_r} \cap S_{\pi_{\ell_r}}) = 0$ for all r. Thus, we have $\dim_H(S_R \cap S_\ell) = 0$. This shows that the set S_δ still has the same threshold property with respect to the behavior of the Hausdorff dimension of the intersection with S_R, as in the case of the individual S_C of a single code.

6.3. Random processes and fractal measures for limit points of codes. We have seen how, for an individual code $C \subset A^n$ we can construct a fractal set S_C of Hausdorff dimension the code rate R and with the Hausdorff measure μ_R in dimension R satisfying the self-similarity condition

$$\mu_R = q^{-nR} \sum_{a \in A^n} \mu_R \circ \sigma_a^{-1}.$$

We now consider the case of a limit point (R, δ), which is an accumulation point of the code domain, so that we have a family of codes C_r with $k_r/n_r \to R$ and $d_r/n_n \to \delta$. As we have seen in Proposition 6.2.1 above, we can construct a set $S_R \subset (0,1)^\infty_q$ with Hausdorff dimension $\dim_H(S_R) = R$.

The construction of S_R shows that the Hausdorff dimension of each S_{C_r} is dominated by that of the larger ones and of S_R. Therefore for the uniform R-dimensional Hausdorff measure each of the S_{C_r} becomes negligible. However, it is possible to construct non-uniform measures on S_R that give non-trivial probability to each of the S_{C_r}. We investigate here how to obtain self-similar multifractal measures on the sets S_R using the method of Ruelle transfer operators.

On the set $S_R \subset (0,1)^\infty_q$ we consider a potential $W = W_\beta$ with non-negative real values satisfying the Keane condition

$$\sum_a W_\beta(ax) = 1, \qquad \forall x \in S_R, \tag{6.2}$$

where for $x \in S_{C_r} \subset S_R$ the sum is over all the elements $a \in C_r$.

The Ruelle transfer operators on S_R will then be of the form

$$\mathcal{R}_{\sigma,W} f(x) = \sum_{\sigma(y)=x} W(y) f(y) = \sum_{a \in \cup_r (C_r \cap A^{n_r})} W(ax) f(ax), \tag{6.3}$$

where the shift map σ on S_R is the one induced by the shift maps on the individual S_{C_r}. The partial inverses of σ are given by maps $\sigma_a(x) = ax$, where, for $x \in S_{C_r}$, a is an element of corresponding C_r.

Example 1. One can consider the case where the potential $W_\beta(x)$ is a piecewise constant function on S_R, which depends only on the first coordinate (first row) $x_1 \in \cup_r (C_r \cap A^{n_r})$ of x. One can write it in this case as

$$W_\beta(x) = e^{-\beta \lambda_{x_1}}, \quad \text{with} \quad \sum_{a \in \cup_r (C_r \cap A^{n_r})} e^{-\beta \lambda_a} = 1. \tag{6.4}$$

Example 2. Another case we will consider in the following is where the potential is also a piecewise constant function on S_R, but which depends on the first two coordinates (first two rows) $x_1, x_2 \in \cup_r (C_r \cap A^{n_r})$ of $x \in (0,1)_q^\infty$. In this case we write it in the form

$$W_\beta(x) = e^{-\beta \lambda_{x_1 x_2}}, \quad \text{with} \quad \sum_{a \in \cup_r (C_r \cap A^{n_r})} e^{-\beta \lambda_{a x_1}} = 1, \tag{6.5}$$

for all $x_1 \in \cup_r (C_r \cap A^{n_r})$. We then think of λ_{ab} as an infinite matrix indexed by elements $a, b \in \cup_r (C_r \cap A^{n_r})$. The condition that $\sum_a W_\beta(ax) = 1$ for all $x \in S_R$ implies that the function $f(x) = 1$ is a fixed point for the transfer operator $\mathcal{R}_{\sigma,W,\beta}$.

Here is a version of the construction given in [DutJor] (see also for instance [MarPa]), for an arbitrary potential W_β satisfying the Keane condition.

6.3.1. Proposition. *For a choice of a point $x_0 \in S_R$, one can then construct a measure $\mu_{\beta x_0}$ on S_R by assigning to the subset $S_R(w) \subset S_R$ of words $x \in S_R$ that start with a given finite length word $w = w_1 \cdots w_m$ with $w_j \in \cup_r (C_r \cap A^{n_r})$ the measure*

$$\mu_{\beta,x_0}(S_R(w)) = W_\beta(w_1 x_0) W_\beta(w_2 w_1 x_0) \ldots W_\beta(w_n \ldots w_1 x_0). \tag{6.6}$$

Proof. To see that this indeed defines a probability measure we need to check that

$$\sum_w \mu_{\beta,x_0}(S_R(w)) = 1,$$

and that
$$\sum_{a\in\cup_n A^n} \mu_{\beta,x_0}(S_R(wa)) = \mu_{\beta,x_0}(S_R(w)).$$

The first condition is satisfied since we have

$$\sum_{w_1\cdots w_n} W_\beta(w_1x_0)W_\beta(w_2w_1x_0)\cdots W_\beta(w_{n-1}\cdots w_1x_0)W_\beta(w_n\cdots w_1x_0) =$$

$$\sum_{w_1\cdots w_{n-1}} W_\beta(w_1x_0)W_\beta(w_2w_1x_0)\cdots W_\beta(w_{n-1}\cdots w_1x_0) = \dots$$

$$= \sum_{w_1} W_\beta(w_1x_0) = 1,$$

by repeatedly using the Keane condition (6.2). The second condition also follows from (6.2), since we have

$$\sum_a \mu_{\beta,x_0}(S_R(wa)) = \sum_a W_\beta(w_1x_0)\cdots W_\beta(w_n\cdots w_1x_0)W_\beta(aw_n\cdots w_1x_0)$$

$$= W_\beta(w_1x_0)W_\beta(w_2w_1x_0)\cdots W_\beta(w_n\cdots w_1x_0),$$

since $\sum_a W_\beta(aw_n\cdots w_1x_0) = 1$.

This completes the proof.

The idea is that one thinks of the measure constructed as above as the probability of a random walk that starts at x_0 and proceeds at each step in the direction marked by an element $a \in \cup_r(C_r \cap A^{n_r})$. In the special cases (6.4) and (6.5), the probabilities are given, respectively, by

$$\mu_{\beta,x_0}(S_R(w)) = \prod_{j=1}^m e^{-\beta\lambda_{w_j}},$$

which is, in this case, independent of the choice of the point x_0, and by

$$\mu_{\beta,x_0}(S_R(w)) = e^{-\beta\lambda_{w_n w_{n-1}}}\cdots e^{-\beta\lambda_{w_2 w_1}} e^{-\beta\lambda_{w_1 x_0}}.$$

Consider then a fixed S_{C_r} inside $S_R = \cup_r S_{C_r}$. The measure constructed as above on S_R induces a multi-fractal measure on each S_{C_r}. We describe the resulting system of measures explicitly in the two cases where the measure on S_R satisfies (6.4) or (6.5).

6.3.2. Proposition. *(1) If the measure on S_R satisfies (6.4), then it induces on each S_{C_r} a multi–fractal measure by assigning*

$$\mu_{\beta,r}(S_{C_r}(w)) = \frac{1}{Z_r(\beta)^m} \prod_j e^{-\beta\lambda_{w_j}}, \tag{6.7}$$

for $w = w_1 \cdots w_m$ with $w_i \in C_r$, where $Z_r(\beta)$ is given by

$$Z_r(\beta) = \sum_{a \in C_r} e^{-\beta\lambda_a}. \tag{6.8}$$

(2) If the measure on S_R satisfies (6.5), then it induces on each S_{C_r} a multi-fractal measure by assigning

$$\mu_{\beta,r,x_0}(S_{C_r}(w)) = \frac{W_\beta(w_m w_{m-1}) \cdots W_\beta(w_1 x_0)\, f^{(r)}_{w_m}}{\rho^m_{\beta,r}\, f^{(r)}_{x_0}}, \tag{6.9}$$

for $w = w_1 \cdots w_m$ with $w_i \in C_r$, where $f^{(r)}$ is the Perron–Frobenius eigenvector of the positive matrix $W_\beta(ab) = e^{-\beta\lambda_{ab}}$ and $\rho_{\beta,r}$ the eigenvalue equal to the spectral radius.

Proof. When one restricts the potential W_β from S_R to a single S_{C_r}, the infinite sum (6.2) is replaced by a truncated finite sum

$$\sum_{a \in C_r \cap A^{n_r}} W_\beta(ax) < 1, \qquad \forall x \in S_R. \tag{6.10}$$

Thus, in the case (6.4), instead of the normalization condition given by the infinite sum

$$\sum_{a \in \cup_r C_r} e^{-\beta\lambda_a} = 1,$$

we have a partition function given by the finite sum (6.8). The induced probability measure on S_{C_r} is then given by assigning measures

$$\mu_{\beta,r}(S_{C_r}(a)) = \frac{e^{-\beta\lambda_a}}{Z_r(\beta)},$$

and more generally by (6.7) on the sets $S_{C_r}(w)$ with $w = w_1 \cdots w_m$ with $w_i \in C_r$. Since $Z_r(\beta)^{-1} \sum_{a \in C_r} e^{-\beta \lambda_a} = 1$, this assignment satisfies the required properties in order to define a probability measure on S_{C_r}. Notice that the measure obtained in this way is no longer a uniform self-similar measure like the Hausdorff measure on S_{C_r} of Hausdorff dimension k_r/d_r, but it is a non-uniform multi-fractal measure in the sense of [Fal], §17.

The case where the potential W_β on S_R satisfies (6.5) is similar. The restriction of W_β to a single S_{C_r} gives a $q^{k_r} \times q^{k_r}$-matrix, $W_\beta(ab) = e^{-\beta \lambda_{ab}}$, for $a, b \in C_r$. This matrix is positive, in the sense that all its entries are, by construction, positive real numbers. Thus, the Perron–Frobenius theorem applied to the matrix $W_\beta(ab)$ (or rather to its transpose) shows that there exists a unique eigenvector $f^{(r)} = (f_a^{(r)})$

$$\sum_{a \in C_r} W_\beta(ab) f_a^{(r)} = \rho_{\beta,r}\, f_b^{(r)}, \tag{6.11}$$

with positive entries $f_a^{(r)} > 0$ and with eigenvalue $\rho_{\beta,r}$ equal to the spectral radius of $W_\beta(ab)$.

We then show that setting the measure of $S_{C_r}(w)$ equal to (6.9), for $w = w_1 \cdots w_m$ with $w_j \in C_r$, defines an induced probability measure on S_{C_r}. We check that

$$\sum_w \mu_{\beta,r,x_0}(S_{C_r}(w)) = \sum_{w_1 \cdots w_m} \frac{W_\beta(w_m w_{m-1}) \cdots W_\beta(w_1 x_0)\, f_{w_m}^{(r)}}{\rho_{\beta,r}^m\, f_{x_0}^{(r)}}$$

$$= \sum_{w_1 \cdots w_{m-1}} \frac{W_\beta(w_{m-1} w_{m-2}) \cdots W_\beta(w_1 x_0)\, f_{w_{m-1}}^{(r)}}{\rho_{\beta,r}^{(m-1)}\, f_{x_0}^{(r)}}$$

$$= \sum_{w_1} \frac{W_\beta(w_1 x_0)\, f_{w_1}^{(r)}}{\rho_{\beta,r}\, f_{x_0}^{(r)}} = 1,$$

since we have

$$\sum_{w_{j+1}} W_\beta(w_{j+1} w_j) f_{w_{j+1}}^{(r)} = \rho_{\beta,r} f_{w_j}^{(r)}.$$

Similarly, we have

$$\sum_a \mu_{\beta,r,x_0}(S_{C_r}(wa)) = \sum_a \frac{W_\beta(a w_m) \ldots W_\beta(w_1 x_0)\, f_a^{(r)}}{\rho_{\beta,r}^{m+1}\, f_{x_0}^{(r)}} =$$

$$\frac{W_\beta(w_m w_{m-1})\cdots W_\beta(w_1 x_0)\, f^{(r)}_{w_m}}{\rho^m_{\beta,r}\, f^{(r)}_{x_0}} = \mu_{\beta,r,x_0}(S_{C_r}(w)),$$

since we have

$$\sum_a W_\beta(a w_m) f^{(r)}_a = \rho_{\beta,r} f^{(r)}_{w_m}.$$

We therefore obtain a family of induced multi-fractal probability measures on the S_{C_r}.

This completes the proof.

A similar construction can be done in the case of the family of sets $S_{\pi_{d_r}}$ with $d_r/n_r \nearrow \delta$ and the set $S_\delta = \cup_r S_{\pi_{d_r}}$.

6.4. Limit points and algebra representations. As above, consider a family of codes C_r with parameters $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$. We have Toeplitz algebras TO_{C_r} associated to each code in this family. It is then natural to consider as algebra associated to the limit point (R,δ) the infinite Toeplitz algebra in the union of the generators of all the TO_{C_r}, namely $TO_{\cup_r C_r}$ generated by isometries S_a for $a \in \cup_r C_r$.

6.4.1. Proposition. *Let μ_{β,x_0} be a probability measure on S_R constructed as above, in terms of a potential $W_\beta(x)$. The algebra $TO_{\cup_r C_r}$ has a representation on the Hilbert space $L^2(S_R, \mu_{\beta,x_0})$ given by*

$$(S_a f)(x) = W_\beta(a x_0)^{-1/2}\, \chi_{S_R(a)}(x)\, f(\sigma(x)), \tag{6.12}$$

for $a \in \cup_r C_r$.

Proof. We must check that the operators (6.12), for $a \in \cup_r C_r$, satisfy the relations $S_a^* S_a = 1$ of $TO_{\cup_r C_r}$, with $S_a S_a^* = P_a$ orthogonal range projections.

First observe that the Radon-Nikodym derivative of μ_{β,x_0} with respect to composition with σ_a for $a \in \cup_r C_r$ satisfies

$$\frac{d\mu_{\beta,x_0}\circ\sigma_a}{d\mu_{\beta,x_0}} = W_\beta(a x_0). \tag{6.13}$$

In fact, we have

$$\mu_{\beta,x_0}(S_R(w)) = \sum_a \mu_{\beta,x_0}(S_R(wa)) =$$

$$\sum_a \int_{S_R(w)} \frac{d\mu_{\beta,x_0}\circ\sigma_a}{d\mu_{\beta,x_0}} d\mu_{\beta,x_0} = \sum_a W_\beta(a x_0)\mu_{\beta,x_0}(S_R(w)).$$

It then follows that the operators S_a of (6.12) have adjoints

$$(S_a^* f)(x) = W_\beta(ax_0)^{1/2} f(\sigma_a(x)). \tag{6.14}$$

In fact, we have

$$\langle S_a h, f\rangle = \int_{S_R(a)} W_\beta(ax_0)^{-1/2}\, \overline{h(\sigma(x))}\, f(x)\, d\mu_{\beta,x_0}(x)$$

$$= \int_{S_R} W_\beta(ax_0)^{-1/2}\, \overline{h(u)}\, f(\sigma_a(u))\, \frac{d\mu_{\beta,x_0}\circ\sigma_a}{d\mu_{\beta,x_0}} d\mu_{\beta,x_0}(u)$$

$$= \int_{S_R} \overline{h(u)}\, W_\beta(ax_0)^{1/2}\, f(\sigma_a(u))\, d\mu_{\beta,x_0}(u) = \langle h, S_a^* f\rangle.$$

One then sees explicitly that the operators S_a and S_a^* satisfy $S_a^* S_a = 1$, while $S_a S_a^*$ is the range projection P_a given by multiplication by the characteristic function $\chi_{S_R(a)}$. Notice that, for $a \neq a'$ in $\cup_r C_r$, the sets $S_R(a)$ and $S_R(a')$ are disjoint, hence the range projections are orthogonal. Thus, we obtain a representation of the algebra $TO_{\cup_r C_r}$.

This completes the proof.

One can proceed in a similar way with respect to the parameter δ using the set S_δ with a similar measure and representation. Thus, the choice of a limit point (R,δ) corresponds to the pair of Hilbert spaces $L^2(S_R, \mu_{\beta,x_0})$ and $L^2(S_\delta, \mu_{\beta',x_0'})$ with representations of the algebras $TO_{\cup_r C_r}$ and $TO_{\cup_r \pi_{d_r}}$, respectively.

The main asymptotic problem of codes ([Man], [TsfaVla]) consists of identifying a continuous curve $R = \alpha_q(\delta)$ (which can also be symmetrically formulated as $\delta = \alpha_q'(R)$) that gives for fixed δ the maximal possible value of R in the closure of the subset of limit points of the code domain (respectively, the maximal δ for fixed R). We describe here a way to characterize the curve $R = \alpha_q(\delta)$ in terms of the measures μ_{β,x_0} on the sets S_R and the uniform self-similar measures on the S_{C_r} for approximating families of codes.

We have shown earlier that given a point (R,δ) in the closure of the code domain, it is always possible to construct an approximating family of codes C_r with $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$. In the following, we refer to such a family $\{C_r\}$ as a *good approximating family.*

We have shown that a measure μ_{β,x_0} on the set $S_R \subset (0,1)_q^\infty$ induces a compatible family of non-uniform fractal measures on the sets $S_{C_r} \subset (0,1)_q^{n_r}$. We now show that, conversely, the family of uniform self-similar measures on the S_{C_r} determine a family of non-uniform measure μ_{β,x_0} on the set $S_R \subset (0,1)_q^\infty$, for $\beta > R$.

6.5. Proposition. *Let C_r be a good approximating family for a limit point (R,δ). For $a \in \cup_r C_r$ set $\lambda_a = n_r \log q$, where n_r corresponds to the smallest $C_r \subset (0,1)_q^{n_r}$ for which $a \in C_r$. Then the series*

$$Z_{\cup_r C_r}(\beta) := \sum_{a \in \cup_r C_r} e^{-\beta\lambda_a} \tag{6.15}$$

converges for $\beta > R$ and the potential

$$W_\beta(x) = Z_{\cup_r C_r}(\beta)^{-1} \exp(-\beta\lambda_{x_1}) \tag{6.16}$$

defines a probability measure on the set S_R. The analogous construction holds for S_δ with convergence in the domain $\beta > \delta$.

Proof. We have

$$Z_{\cup_r C_r}(\beta) = \sum_r q^{k_r} q^{-\beta n_r},$$

since the S_{C_r} are disjoint in $(0,1)_q^\infty$. Since $\{C_r\}$ is a good approximating family, we have $k_r/n_r \leq R$ and we see that

$$\sum_r q^{k_r} q^{-\beta n_r} \leq \sum_r q^{(R-\beta)n_r}.$$

This is convergent for $\beta > R$. The potential $W_\beta(x)$ of (6.16) then satsifies the Keane condition $\sum_a W_\beta(ax) = 1$. The construction for S_δ is entirely analogous, using the uniform measures on the $S_{\pi_{d_r}}$. This completes the proof.

We then obtain the following characterization of the curve $R = \alpha_q(\beta)$ of the fundamental asymptotic problem for codes.

6.6. Proposition. *The domain $\beta \geq \alpha_q(\delta)$ is the closure of the common domain of convergence of the functions $Z_{\cup_r C_r}(\beta)$ for all the points (R,δ) with fixed δ in the closure of the subset of limit points of the code domain and for all good approximating families $\{C_r\}$.*

Proof. The domain $\beta \geq R$ is in fact the closure of the common domain of convergence of the functions $Z_{\cup_r C_r}(\beta)$ when one varies the good approximating family C_r. In fact, the argument above shows that they all converge for $\beta > R$. The

S_{C_r} are disjoint in $(0,1)_q^\infty$ so that the zeta function (6.15) is given by $\sum_r q^{k_r} q^{-\beta n_r}$. Then if $\beta < R$, for sufficiently large r one will have $k_r/n_r - \beta > 0$ and the series diverges. Then by varying the limit point (R,δ) with fixed δ one obtains the result.

Remark. We constructed in §6.4 multi-fractal measures on the set $\cup_r S_{C_r}$ for a family of codes $\{C_r\}$ approximating a limit point (R,δ). We also considered, associated to the same family of codes, the infinite Toeplitz algebra $TO_{\cup_r C_r}$. Notice that in this case, unlike what happens for the case of a single code, the set $\cup_r S_{C_r}$ is no longer dense in the spectrum of the maximal abelian subalgebra. In fact, the latter consists of all infinite sequences in the elements of $\cup_r C_r$, while the set $\cup_r S_{C_r}$ only contains those sequences where all the successive elements in an infinite sequence belong to the same C_r. Both sets can be regarded as the union of the ω-languages defined by the codes C_r, where in the case of $\cup_r S_{C_r}$ one is keeping track of the information of the embeddings of the codes $C_r \subset A^{n_r}$, that is, of viewing elements of each language as matrices so that the concatenation operation of successive words can only happen for matrices that has the same row lengths, while in the case of the spectrum of the maximal abelian subalgebra one does not take the embedding into account so that all concatenations of words in the languages defined by the codes C_r are possible and one obtains a larger set.

6.7. Quantum statistical mechanics above and below the asymptotic bound. We have seen in §4 how to associate a quantum statistical mechanical system to an individual code. We also know from Theorem 2.10 that code points have multiplicities: in particular, code points that lie below the asymptotic bound have infinite multiplicity, while isolated codes, which lie above the asymptotic bound have finite multiplicity. In terms of quantum statistical mechanical systems, it is therefore more natural to fix a code point (R,δ) and construct an algebra with time evolution $(TO_{(R,\delta)},\sigma)$ which does not depend on choosing a code C representing the code point, but allowing for all representatives simultaneously. This can be done in the same way we used in §6.4 for limit points. Namely, we let $TO_{(R,\delta)}$ be the Toeplitz algebra with generators the elements in the union of all codes C with parameters (R,δ). This will be isomorphic to a finite rank Toeplitz algebra TO_N for isolated codes and isomorphic to the infinite Toeplitz algebra TO_∞ in the case of code points that lie below the asymptotic bound. Similarly, we can consider the fractal set given by the union of the S_C for all the representative codes with fixed (R,δ). In this case all these sets have the same Hausdorff dimension equal to R, but in the case of isolated codes they are obtained as a finite union and therefore they admit a uniform self-similar probability measure, the R-dimensional Hausdorff measure, while in the case of the points below the asymptotic bound one can construct non-uniform probability measure using the same method we described in §6.2 for limit points. We can use potentials as in (6.4) to construct such measures. This in turn induces a time evolution on $TO_{(R,\delta)}$ of the form

$$\sigma_t(T_a) = e^{it\lambda_a}\, T_a.$$

In this way, the properties of the quantum statistical mechanical system associated to a code point (R,δ) reflect the difference between point above or below the asymptotic bound.

7. The asymptotic bound as a phase diagram.

The goal of this section is to extend the construction of quantum statistical mechanical systems from the case of individual codes C to families of codes in such a way as to obtain a description of the asymptotic bound $R = \alpha_q(\delta)$ as a phase transition curve in a phase diagram.

7.1. Variable temperature KMS states. We begin by giving here a generalization of the usual notion of KMS states, which we refer to as *variable temperature KMS states* and which will be useful in our example. This is similar to the notion of "local KMS states" considered, for instance, in [Acca] in the context of out of equilibrium thermodynamics, as well as in the context of information theory in [InKoO], though definition we give here is more general. We formulate it first in the case of an arbitrary algebra of observables and we then specialize it to the case of families of codes.

7.7.1. Definition. *Let $\mathcal{B}$ be a unital C^*-algebra and let $\mathcal{X}$ be a parameter space, assumed to be a (compact Hausdorff) topological space, together with an assigned continuous function $\beta : \mathcal{X} \to \mathbf{R}_+$. For $t \in C(\mathcal{X}, \mathbf{R})$, let $\sigma_t \in \mathrm{Aut}(\mathcal{B})$ be a family of automorphisms satisfying $\sigma_{t_1+t_2} = \sigma_{t_1} \circ \sigma_{t_2}$. A KMS$_\beta$ state for $(\mathcal{B}, \sigma)$ is a continuous linear functional $\varphi : \mathcal{B} \to \mathbf{C}$ with $\varphi(1) = 1$ and $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{B}$, and such that, for all $a, b \in \mathcal{B}$ there exists a function $F_{a,b}(z)$, for $z : \mathcal{X} \to \mathbf{C}$, with the property that the function $F_{a,b}(z(\alpha))$ for any fixed $\alpha \in \mathcal{X}$ and varying $z \in C(\mathcal{X}, \mathbf{C})$ is a holomorphic function of the complex variable $z(\alpha) \in I_{\beta(\alpha)}$, where*

$$I_{\beta(\alpha)} = \{z \in \mathbf{C} \,|\, 0 < \Re(z) < \beta(\alpha)\},$$

and extends to a continuous function on the boundary of $I_{\beta(\alpha)}$ with

$$F_{a,b}(t(\alpha)) = \varphi(a\sigma_{t(\alpha)}(b)), \qquad \text{and} \qquad F_{a,b}(t(\alpha) + i\beta(\alpha)) = \varphi(\sigma_{t(\alpha)}(b)a),$$

where $t(\alpha) = z(\alpha)|_{\Re(z(\alpha))=0}$.

Example. In the case where the parameter space is a finite set of points, say $\mathcal{X} = \{1, \dots, N\}$ one finds that σ_t is an action of $\mathbf{R}^N$ by automorphisms and the variable temperature KMS condition gives a functional such that $\varphi(ab) = \varphi(\sigma_{i\beta}(b)a)$, with $\beta = (\beta_1, \dots, \beta_N)$. The partition function, correspondingly, is a function $Z(\beta_1, \dots, \beta_N) = \mathrm{Tr}(e^{-\langle \beta, H\rangle})$, where $H = (H_k)$ implements the time evolution σ_t in the sense that

$$\pi(\sigma_t(a)) = e^{i\langle t,H\rangle}\pi(a)e^{-i\langle t,H\rangle},$$

in a given Hilbert space representation π of $\mathcal{B}$.

We are interested in the case where the algebra is itself a tensor product over the parameter space, and the resulting C^*-dynamical system is also a tensor product. Namely, we have $\mathcal{B} = \otimes_\alpha \in \mathcal{X}\mathcal{B}_\alpha$ with $\sigma_t = \otimes_\alpha \sigma_{t(\alpha)}$ and a representation $\pi = \otimes_\alpha \pi_\alpha$ on a product $\mathcal{H} = \otimes_\alpha \mathcal{H}_\alpha$, with a Hamiltonian $H = \otimes_\alpha H_\alpha$ generating the time evolution, namely so that on $\mathcal{H}_\alpha$ one has

$$\pi_\alpha(\sigma_{t(\alpha)}(a_\alpha)) = e^{it(\alpha)H_\alpha}\pi_\alpha(a_\alpha)e^{-it(\alpha)H_\alpha}.$$

Then for a given $\beta : \mathcal{X} \to \mathbf{R}_+$, a state $\varphi = \otimes_\alpha \varphi_\alpha$ is a KMS_β state iff the φ_α are $\mathrm{KMS}_{\beta(\alpha)}$ states for the time evolution $\sigma_{t(\alpha)}$. We assume here that $\mathcal{X}$ is a discrete set and that the C^*-algebras $\mathcal{B}_\alpha$ are nuclear so that tensor products over finite subsets of $\mathcal{X}$ are unambiguously defined and the product over $\mathcal{X}$ is obtained as direct limit, as in Proposition 7 of [BoCo].

7.2. Phase transitions for families of codes. We consider approximations to the curve $R = \alpha_q(\delta)$ by families of N points (δ_j, R_j) that are code points, that is, for which there exists a code C_j with $k_j/n_j = R_j$ and $d_j/n_j = \delta_j$. To such a collection of points we associate a quantum statistical mechanical system that is the tensor product of the systems associated to each code C_j, with algebra of observables $\mathcal{A} = \otimes_j TO_{C_j}$ and with the dynamics given by $\sigma : \mathbf{R}^N \to \mathrm{Aut}(\mathcal{A})$, with $\sigma_t = \otimes_j \sigma_{t_j}$, where σ_{t_j} is the time evolution on TO_{C_j} given by

$$\sigma_{t_j}(S_a) = q^{itn_j}S_a.$$

7.2.1. Lemma. *Let $(\mathcal{A}, \sigma)$ be the product system described above, for a collection C_j of codes, with $j = 1, \dots, N$. Then for any given $\beta = (\beta_1, \dots, \beta_j)$ there is a unique KMS_β state on $(\mathcal{A}, \sigma)$, which is given by the product $\varphi_\beta = \otimes_j \varphi_{\beta_j}$ of the unique KMS_{β_j} states on the algebras TO_{C_J}. For β in the region $\beta_j > R_j$, the KMS* *state is of type I_∞. The partition function is the product of the partition functions of the individual systems.*

Proof. The product state $\varphi_\beta = \otimes_j \varphi_{\beta_j}$ is a KMS$_\beta$ state for $(\mathcal{A}, \sigma)$ with $\beta = (\beta_1, \dots, \beta_j)$. The uniqueness for the tensor product state follows from an argument similar to the one used in Proposition 8 of [BoCo], adapted to our more general notion of KMS state. It suffices in fact to observe that, if φ is a KMS$_\beta$ state with $\beta = (\beta_1, \dots, \beta_j)$ on the product $\mathcal{A} = \otimes_j TO_{C_j}$, then for fixed $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N$, the functional

$$\varphi_{a_1 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \otimes \cdots \otimes a_N}(a_j) = \varphi(a_1 \otimes \cdots \otimes a_N)$$

is a KMS$_{\beta_j}$ state on TO_{C_j}, by the same argument used in the ordinary case.

The Hamiltonian H_j generating the time evolution σ_{t_j} on the algebra TO_{C_j} has eigenvalues $mn_j \log q$, with integers $m \geq 0$, with multiplicities q^{mk_j}, and partition function

$$Z(\beta_j) = \mathrm{Tr}(e^{-\beta_j H_j}) = \sum_m q^{(R_j - \beta_j) n_j m} = (1 - q^{(R_j - \beta_j) n_j})^{-1}.$$

The partition function for the product system is then

$$Z(\beta_1, \dots, \beta_N) = \mathrm{Tr}(e^{-\sum_j \beta_j H_j}) = \sum_{m = (m_1, \dots, m_N)} q^{\sum_j (R_j - \beta_j) n_j m_j}$$

$$= \prod_j \Big(\sum_{m_j} q^{(R_j - \beta_j) n_j m_j}\Big) = \prod_j (1 - q^{(R_j - \beta_j) n_j})^{-1} = \prod_j Z(\beta_j).$$

It converges in the domain of $\mathbf{R}^N$ determined by the conditions $\beta_j > R_j$.

This finishes the proof.

To further refine the picture described above, we consider quantum statistical mechanical systems associated to families of codes approximating a limit point in the closure of the code domain.

As before, let $\mathcal{C} = \{C_r\}$ be a family of codes with $k_r/n_r \nearrow R$ and $d_r/n_r \nearrow \delta$. We consider again the union $\cup_r C_r$ and the corresponding Toeplitz algebra $TO_{\cup_r C_r}$. On the fractal $S_R = \cup_r S_{C_r}$ of Hausdorff dimension $\dim_H(S_R) = R$, consider a potential $W_\beta(x) = e^{-\beta \lambda_{x_1}}$, such that, when $\beta = R$ it satsifies the Keane condition

$$\sum_{a \in \cup_r C_r} e^{-R\lambda_a} = 1.$$

We consider then the time evolution on $TO_{\cup_r C_r}$ given by

$$\sigma_t^W(T_a) = e^{it\lambda_a} T_a.$$

In the representation of $TO_{\cup_r C_r}$ on its Fock space, this time evolution is generated by a Hamiltonian

$$H\epsilon_w = (\lambda_{w_1} + \cdots + \lambda_{w_m})\epsilon_w,$$

for $w = w_1 \cdots w_m$ with $w_i \in \cup_r C_r$. This has partition function

$$Z_{\mathcal{C}}(\beta) = \sum_m \sum_{w \in \mathcal{W}_{\cup_r C_r, m}} e^{-\beta(\lambda_{w_1} + \cdots + \lambda_{w_m})} = \sum_m \left(\sum_{a \in \cup_r C_r} e^{-\beta \lambda_a} \right)^m .$$

If we introduce the notation

$$\Lambda(\beta) := \sum_{a \in \cup_r C_r} e^{-\beta \lambda_a},$$

we have $\Lambda(R) = 1$ and, for $\beta > R$, $\Lambda(\beta) < 1$, while for $\beta < R$ one has $\Lambda(\beta) > 1$, which becomes possibly divergent after some critical value $\beta_0 < R$. Thus, the partition function for the system $(TO_{\cup_r C_r}, \sigma^W)$ is

$$Z_{\mathcal{C}}(\beta) = \sum_m \Lambda(\beta)^m = (1 - \Lambda(\beta))^{-1},$$

convergent for $\beta > R$, with a phase transition at $\beta = R$. The same argument of Proposition 4.7.3 can be extended to this case to show the existence at all $\beta > 0$ of a unique KMS$_\beta$ state, which is of type I_∞ below the critical temperature and is given by a residue at the critical temperature.

One can then consider approximations of the curve $R = \alpha_q(\delta)$ by points $(R_{j,N}, \delta_{j,N})$ in U_q, for $j = 1, \ldots, N$. To each of these points one associates a quantum statistical mechanical system constructed as above using a family $\mathcal{C}_{j,N} = \{C_{r_{j,N}}\}$ of codes approximating the limit point $(\delta_{j,N}, R_{j,N})$ with the time evolution $\sigma^{W_{j,N}}$ described above on the algebra $TO_{\mathcal{C}_{j,N}}$. By taking the product of these systems one can form a system with variable temperature KMS states with phase transition at $\beta_{j,N} = R_{j,N} \leq \alpha_q(\delta_{j,N})$. This can be extended to the case of a countable dense set of points below the curve $R = \alpha_q(\delta)$ and the corresponding countable tensor product system.

It would be interesting to extend this type of tensor product construction for families of algebras associated to codes to a version that corresponds to a "system with interaction" more like the Bost–Connes algebra.

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