FROBENIUS MANIFOLD STRUCTURES ON THE SPACES OF ABELIAN INTEGRALS

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Abstract. Frobenius manifold structures on the spaces of abelian integrals were constructed by I. Krichever. We use $\mathcal{D}$-modules, deformation theory, and homological algebra to give a coordinate-free description of these structures. It turns out that the tangent sheaf multiplication has a cohomological origin, while the Levi–Civita connection is related to one-dimensional isomonodromic deformations.

1. Introduction

Frobenius manifolds are manifolds with a flat metric and a multiplication in the tangent sheaf, subject to some constraints. Frobenius manifolds were introduced by B. Dubrovin in [D1, D2] as a mathematical framework for deformations of topological quantum field theories (see also [D3]). In mathematics Frobenius manifolds arise in two different situations, corresponding to A-models and B-models in physics. In an A-model one counts rational curves on a variety; this is also known as Gromov–Witten invariants. The generating function for these invariants is the potential for the corresponding Frobenius manifold.

This paper is concerned with B-models. In a B-model one studies deformations of a certain complex structure (formal or analytic). The best known examples are extended moduli spaces of Calabi–Yau varieties [BK] and the unfoldings of isolated singularities [Sai] (see [Sab] for an exposition). We would like to mention that Frobenius structures are important for mirror symmetry: if two varieties are mirror dual to each other, then the A-model Frobenius manifold, corresponding to the first variety, is isomorphic to the B-model Frobenius manifold, corresponding to the second.

1.1. **Moduli spaces of abelian integrals.** Examples of Frobenius manifolds are furnished by Hurwitz spaces. Hurwitz spaces parameterize pairs (X, f), where X is a smooth complete algebraic curve, $f : X \to \mathbb{P}^1$. Dubrovin constructed Frobenius structures on Hurwitz spaces [D3].

Our main object is the following deformation of a Hurwitz space: a space of pairs (X, f), where f is a *multi-valued* function such that df is a single-valued meromorphic 1-form with prescribed periods and residues. If the periods and residues are equal to zero, then this space is a Hurwitz space. Our spaces will be called *spaces of abelian integrals*.

Krichever constructs in [K1, K2] Frobenius structures on the universal covers of the spaces of abelian integrals. Our main goal is to give a coordinate-free geometric description of these Frobenius structures. We also generalize the setup to the case of multiple poles and non-zero residues in §5. In particular, our generalization covers the previously untreated case of abelian integrals of the third kind. Our approach is based on a $\mathcal{D}$-module push-forward (also known as twisted de Rham complex; see [Sab, §I.3.3]). It turns out that these structures of Frobenius manifolds have a nice interpretation: the tangent sheaf multiplication has a cohomological origin, similar to that of [BK]. The metric and the Levi–Civita connection are closely related to one-dimensional isomonodromic deformation. (This is not directly related to isomonodromic deformations used to describe the semi-simple Frobenius manifolds.)

We are using the approach to Frobenius structures via primitive forms. This has been invented by K. Saito [Sai]. We would like to mention a striking similarity between three constructions of Frobenius structures: on the universal unfolding of isolated singularity [Sai], on the extended moduli space of Calabi–Yau varieties [BK], and our construction. In each case a pencil of connections is obtained by the (derived) direct image. Our case is, in some sense, intermediate: on the one hand, singularities are present, on the other hand, our structure is not local, it depends on the global geometry of a curve. This is why we hope that, generalizing our construction to higher dimension, we shall provide a bridge between the pictures of Saito and Barannikov–Kontsevich, giving a unified approach to B-model Frobenius manifolds.

Another interesting feature of our construction is that we get a *family* of Frobenius manifolds parameterized by the periods of abelian integrals. We also want to emphasize that there are some new features specific for the higher genus case: to get a Frobenius structure we need to make a modification of the direct image (see §3.3).

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2. Preliminaries and the main construction

2.1. Pencils of connections.

Definition 1. Let $p_1 : \mathbf{M} \times \mathbb{P}^1 \to \mathbf{M}$ be the natural projection, where $\mathbf{M}$ is a complex manifold, $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}}$. By a *pencil of connections* on $\mathbf{M}$ we mean a pair $(\mathcal{W}, \nabla)$, where $\mathcal{W} \to \mathbf{M}$ is a vector bundle, ∇ is a relative flat connection on $\mathcal{V} = p_1^*\mathcal{W}$ along $\mathbf{M}$ with a simple pole along $\mathbf{M} \times \{0\}$.

One interprets a pencil as a family of flat connections on $\mathcal{W}$, parameterized by $\mathbb{P}^1 \setminus 0$. The condition on the pole implies that this family is of the form $\nabla_\infty + \Phi/z$, this is why it is called "pencil" (here z is a coordinate on $\mathbb{P}^1$). There is a natural way to construct twisted Frobenius manifold structures on dense open subsets of $\mathbf{M}$ starting from a pencil of connections, provided this pencil of connections satisfies some non-degeneracy condition (and, conversely, every Frobenius manifold gives rise to a pencil of connections). This will be explained in details in §4.1.

2.2. Main objects. Consider a smooth complete algebraic curve X of genus g over $\mathbb{C}$, let $p \in X$. Denote by $(\hat{X}, \hat{p})$ the maximal abelian cover of (X, p).

Definition 2. An *abelian integral* on (X, p) is a function f on $\hat{X}$ such that df descends to a meromorphic differential on X. We define *periods* of f to be those of df.

Remarks. (1) An abelian integral can be thought as an integral of a meromorphic form on X. Thus the space of abelian integrals is a one-dimensional affine bundle over the vector bundle of differential forms.

(2) One can avoid working with abelian integrals by fixing a point $p_0 \in \hat{X}$. Then the affine bundle trivializes, a section being the space of abelian integrals that vanish at p_0, see Remark in §5.

To simplify notation, we shall assume first that df has a single pole. We outline the changes needed in the multi-pole case in §5.

Let $n \geq 1$ be an integer. Consider the moduli space $\mathbf{A}_{g,n}$ of triples (X, p, f), where (X, p) is as above, f is an abelian integral with a single pole of order exactly n at p (in other words, df is a meromorphic form on X with the only pole at p of order $n+1$).

The periods of f give a linear map $H_1(X, \mathbb{Z}) \to \mathbb{C}$. One can identify groups $H_1(X, \mathbb{Z})$ locally over the moduli space of curves using the Gauss–Manin connection, therefore the periods give rise to a foliation on $\mathbf{A}_{g,n}$. Let us fix one of the leaves and denote its smooth locus by $\mathbf{A}$. Thus, roughly speaking, $\mathbf{A}$ parameterizes abelian integrals with prescribed periods.

Let $\hat{\mathbf{A}}$ be the moduli space of quadruples (X, p, f, Δ), where $(X, p, f) \in \mathbf{A}$, Δ is a subgroup of $H_1(X, \mathbb{Z})$ maximal isotropic with respect to the intersection form. The elements of Δ will be called *a-cycles.* Clearly, $\hat{\mathbf{A}}$ is a cover of $\mathbf{A}$. Our main result is the following

Theorem. (a) *There is a natural pencil of connections on* $\hat{\mathbf{A}}$.
(b) *This pencil of connections gives rise to a twisted Frobenius structure on* $\hat{\mathbf{A}}$.

Let us comment on the second part. To construct a Frobenius manifold structure from the pencil of connections one needs a so-called *primitive section*, which exist only locally. Another problem is that they are not canonical (see §4.1). It means that $\hat{\mathbf{A}}$ is covered by "Frobenius charts". The tangent sheaf multiplication is nevertheless canonical. This is what we mean by a twisted Frobenius structure. It will be discussed in more details in §4.1. We shall further discuss the choice of a primitive section in §6.

Remarks. (1) We could have worked with fibrations rather than foliations by fixing a basis in $H_1(X, \mathbb{Z})$ (that is, a level structure on the curve). Then the periods give a global map to $\mathbb{C}^{2g}$ and our moduli spaces are the fibers of this map.

(2) $\mathbf{A}$ is not algebraic, so we shall work in analytic category. Also, we can construct a Frobenius structure on the formal completion of $\mathbf{A}$ at a point. However, we wanted to emphasize that our construction is global rather than formal.

2.3. **Notation.** The following notation will be fixed throughout the paper. Let $\varphi : \mathbf{X} \to \mathbf{A}$ be the universal curve, that is, the fiber of φ over (X, p, f) is X. We denote by d_X the relative differential $\mathcal{O}_{\mathbf{X}} \to \Omega_{\mathbf{X}/\mathbf{A}}$. We denote by $d : \mathcal{O}_{\mathbf{X}} \to \Omega_{\mathbf{X}}$ the usual (absolute) differential.

There is a natural section of φ corresponding to $p \in X$, denote it by $\tilde{p}$; we can also view it as a divisor on $\mathbf{X}$. For any integer number k set $\mathcal{O}(k) = \mathcal{O}_{\mathbf{X}}(k\tilde{p})$, $\Omega(k) = \Omega_{\mathbf{X}/\mathbf{A}} \otimes \mathcal{O}(k)$, $\mathcal{T}(k) = \mathcal{T}_{\mathbf{X}/\mathbf{A}} \otimes \mathcal{O}(k)$, where $\Omega_{\mathbf{X}/\mathbf{A}}$ is the sheaf of relative differentials, $\mathcal{T}_{\mathbf{X}/\mathbf{A}}$ is the relative tangent sheaf.

We have a "universal" multi-valued function on $\mathbf{X}$. We denote it again by f, hopefully it will not lead to a confusion. Set $\omega = d_X f$.

2.4. **Main construction.** The fact that the periods of f are fixed shows that df is a single-valued 1-form on $\mathbf{X}$. Thus

$$\nabla = d + \frac{df}{z}$$

is a family of flat connections on $\mathcal{O}_{\mathbf{X}}$ parameterized by $z \in \mathbb{P}^1 \setminus 0$. We can view ∇ as a relative $\mathcal{D}$-module on $\mathbf{X} \times (\mathbb{P}^1 \setminus 0)$. The idea is that we get a pencil of connections on $\mathbf{A}$ by taking the push-forward of ∇ along $\varphi \times \mathrm{Id}_{\mathbb{P}^1}$. There are two problems we shall have to go around

(1) Our $\mathcal{D}$-module is not defined at $z = 0$, and the push-forward is not coherent near $z = \infty$. Thus some regularization is needed.

(2) We get a vector bundle on $\mathbf{A} \times \mathbb{P}^1$, whose restriction to $\{m\} \times \mathbb{P}^1$ is not trivial, thus it is not a pencil of connections. We shall make some modification along $\mathbf{A} \times \{\infty\}$, this is where we need the additional structure of a-cycles.

We shall denote $\varphi \times \mathrm{Id}_{\mathbb{P}^1}$, $\varphi \times \mathrm{Id}_{\mathbb{P}^1 \setminus 0}$ etc. again by φ for brevity.

Remarks. (1) For a fixed z we can start with a connection $d_X + \frac{\omega}{z}$ along the fibers of φ. Then the condition that the periods of ω are constant is exactly the isomonodromic condition for this connection. Thus it can be extended to an absolute connection. See §4.2 for more on isomonodromic deformation.

(2) If f has zero periods (that is, f is a meromorphic function on X), then one can extend ∇ to an absolute flat meromorphic connection on $\mathbf{X} \times \mathbb{P}^1$. In this case we get a Frobenius manifold with Euler field (that is, with a conformal structure). For details see, e.g. [Sab].

(3) The $\mathcal{D}$-module push-forward can be viewed as taking the cohomology fiberwise, with the connection on cohomology being the Gauss–Manin connection. One can also think about this as a de Rham complex, twisted by e^f; see, e.g. [Sab, §I.3.3].

2.5. **Organization of the paper.** In the next section we shall make the ideas above precise, thus constructing a pencil of connections on $\hat{\mathbf{A}}$. In §4 we prove that the pencil of connections gives rise to Frobenius structures on some open subsets of $\hat{\mathbf{A}}$ and calculate these Frobenius structures explicitly, then we present the relation between our construction and that of [K1, K2]. In §5 we generalize our setup to the multi-pole case. In §6 we discuss the choices involved in the construction of Frobenius manifolds from the pencil of connections.

3. The precise construction

Consider the complex

$$\mathcal{O}(-1) \xrightarrow{d_X + \frac{\omega}{z}} \Omega(n).$$

We always place the leftmost term in degree zero. We can view this complex as a complex of sheaves on $\mathbf{X}$, depending on a parameter $z \in \mathbb{P}^1 \setminus 0$. We can also view it as a complex of sheaves of $\mathbf{X} \times (\mathbb{P}^1 \setminus 0)$-modules $\mathcal{O}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1 \setminus 0} \xrightarrow{d_X + \omega/z} \Omega(n) \boxtimes \mathcal{O}_{\mathbb{P}^1 \setminus 0}$.

To regularize the complex at $z = 0$ we shall patch it on $\mathbb{P}^1 \setminus \{0, \infty\}$ with the complex $\mathcal{O}(-1) \xrightarrow{zd_X+\omega} \Omega(n)$, using the diagram

$$\begin{array}{ccc} \mathcal{O}(-1) & \xrightarrow{zd_X+\omega} & \Omega(n) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \times\frac{1}{z}} \\ \mathcal{O}(-1) & \xrightarrow{d_X+\frac{\omega}{z}} & \Omega(n) \end{array} \tag{1}$$

This is an isomorphism of complexes if $z \neq 0, \infty$. This construction gives a complex

$$\mathcal{O}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{d_X+\frac{\omega}{z}} \Omega(n) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1). \tag{2}$$

We shall denote this complex by $\mathcal{D}^\bullet$ and its restriction to $\mathbf{X} \times \{z\}$ by $\mathcal{D}^\bullet_z$.

Note that the map is $\mathcal{O}_{\mathbf{A}\times\mathbb{P}^1}$-linear. We shall view this complex as an object in the derived category of complexes of sheaves of $\varphi^\cdot\mathcal{O}_{\mathbf{A}\times\mathbb{P}^1}$-modules on $\mathbf{X} \times \mathbb{P}^1$. Here $\varphi^\cdot$ is the sheaf-theoretic inverse image. We shall abuse the language by saying "complex of sheaves", where we really mean "complex of sheaves up to a quasi-isomorphism". The push-forward $\varphi_*\mathcal{D}^\bullet$ is a complex of coherent sheaves (because φ is proper).

3.1. **Relation to $\mathcal{D}$-modules.** Denote by $\mathcal{O}(\infty)$ the sheaf of meromorphic functions on $\mathbf{X}$ with poles on $\tilde{p}$ of any order. The connection $d + \frac{df}{z}$ equips $\mathcal{O}(\infty)$ with a structure of z-dependent $\mathcal{D}_{\mathbf{X}}$-module ($z \neq 0$).

To calculate its φ-push-forward we have to consider the corresponding relative de Rham complex

$$\mathcal{O}(\infty) \xrightarrow{d_X+\frac{\omega}{z}} \Omega(\infty). \tag{3}$$

It will be more convenient for us to work with coherent sheaves, so we take a subcomplex of coherent sheaves

$$\mathcal{O}(k) \xrightarrow{d_X+\frac{\omega}{z}} \Omega(k+n+1).$$

One easily checks that it is quasi-isomorphic to (3) at $z \neq \infty$ for any $k \in \mathbb{Z}$. It follows that $\varphi_*\mathcal{D}^\bullet$ has a $\mathcal{D}_{\mathbf{A}}$-module structure for $z \neq 0, \infty$ (because it is isomorphic to a push-forward of a $\mathcal{D}$-module). We shall calculate this structure and see that it extends to $z = \infty$ and has a simple pole at $z = 0$.

Our choice of $k = -1$ is imposed by the fact that in this case the push-forward is locally free, as we shall see shortly. If, for example, we take $k = 0$, the push-forward would have torsion at $z = \infty$ and would not be concentrated in a single homological dimension.

3.2. **Study of $\varphi_*\mathcal{D}^\bullet$ in the direction of $\mathbb{P}^1$.** Now we choose a point $m = (X, p, f_0) \in \mathbf{A}$ and study the restriction of $\varphi_*\mathcal{D}^\bullet$ to $\{m\} \times \mathbb{P}^1$. We denote this restriction again by $\varphi_*\mathcal{D}^\bullet$ for brevity. In this and in the next subsections $\mathcal{O}(k)$ stands for $\mathcal{O}_X(kp)$ and $\Omega(k) = \Omega_X(kp)$.

Lemma 1. *$\varphi_*\mathcal{D}^\bullet$ is isomorphic to the vector bundle*

$$(\oplus_1^g \mathcal{O}_{\mathbb{P}^1}) \oplus \left(\oplus_1^{g+n-1} \mathcal{O}_{\mathbb{P}^1}(1)\right) \tag{4}$$

placed in degree 1.

Proof. Let us show first that $\varphi_*\mathcal{D}^\bullet$ is a vector bundle concentrated in degree 1. To this end we fix $z \neq 0$ and consider the exact sequence

$$0 \to \Omega(n)[1] \to \mathcal{D}_z^\bullet \to \mathcal{O}(-1) \to 0. \tag{5}$$

Let us write the corresponding exact sequence of hypercohomology

$$0 \to \mathbb{H}^0(\mathcal{D}_z^\bullet) \to 0 \to H^0(\Omega(n)) \to \mathbb{H}^1(\mathcal{D}_z^\bullet) \to H^1(\mathcal{O}(-1)) \to 0 \to \mathbb{H}^2(\mathcal{D}_z^\bullet) \to 0.$$

We see that the hypercohomology of $\mathcal{D}_z^\bullet$ are concentrated in degree 1 and the dimension does not depend on z (it is equal to $2g+n-1$). A similar sequence for the upper complex in (1) shows that a similar statement is valid near $z = 0$. Now a standard argument involving a base change shows that $\varphi_*\mathcal{D}^\bullet$ is a locally free sheaf in degree 1, that is, a vector bundle.

Evaluating the global sections of (2) along $\mathbb{P}^1$ first, we come to the following presentation of the global sections of $\varphi_*\mathcal{D}^\bullet$

$$\mathbb{H}^1(\mathcal{O}(-1) \xrightarrow{(d_X, \times\omega)} \Omega(n) \oplus \Omega(n)).$$

Using an exact sequence, similar to (5), one checks easily that the dimension of the space of global sections of $\varphi_*\mathcal{D}^\bullet$ is equal to $3g + 2n - 2 = \dim \varphi_*\mathcal{D}^\bullet + (g + n - 1)$. The lemma will be proved if we show that (i) the global sections generate the fiber of $\varphi_*\mathcal{D}^\bullet$ at $z = 0$ and (ii) that there are no global sections, vanishing at both $z = 0$ and $z = \infty$. Indeed, every vector bundle on $\mathbb{P}^1$ is isomorphic to $\oplus_i \mathcal{O}_{\mathbb{P}^1}(m_i)$. Now (i) shows that $m_i \geq 0$ for all i, and (ii) shows that $m_i \leq 1$ for all i.

It follows from the base change that the following map of complexes gives rise to the map from the space of global sections to $\varphi_*\mathcal{D}^\bullet|_{z=0}$

$$\begin{array}{ccc} \mathcal{O}(-1) & \xrightarrow{(d_X, \times\omega)} & \Omega(n) \oplus \Omega(n) \\ \downarrow & & \downarrow{\scriptstyle (0,1)} \\ \mathcal{O}(-1) & \xrightarrow{\omega} & \Omega(n). \end{array} \tag{6}$$

The second hypercohomology group of the kernel of this map is isomorphic to $H^1(\Omega(n)) = 0$. Hence the induced map on hypercohomology groups is surjective, and (i) is satisfied.

We also see that the kernel of the map from the space of global sections to $\varphi_*\mathcal{D}^\bullet|_{z=0}$ is given by the global sections of the *first* $\Omega(n)$ summand.

Now, one writes a map of complexes, analogous to (6) but generating a map from the space of global sections to $\varphi_*\mathcal{D}^\bullet|_{z=\infty}$ and checks that its kernel is given by the global sections of the *second* $\Omega(n)$ summand. Since these subspaces of global sections do not intersect, (ii) follows. □

Below we often view $\varphi_*\mathcal{D}^\bullet$ as a vector bundle, making the shift by -1 implicit.

Let us denote by $Conn^0_{X,p,n}$ the moduli space of degree zero line bundles on X with a connection such that the connection has a pole of order at most n at p and no other poles. Denote by $\widetilde{Conn}^0_{X,p,n}$ the universal cover of $Conn^0_{X,p,n}$. Note that $Conn^0_{X,p,n}$ is an affine bundle over Pic^0_X.

Lemma 2. *There is a natural isomorphism*

$$\varphi_*\mathcal{D}^\bullet|_{z=\infty} \approx \widetilde{Conn}^0_{X,p,n}. \tag{7}$$

Proof. By the base change LHS is given by $\mathbb{H}^1(\mathcal{O}(-1) \xrightarrow{d} \Omega(n))$. It is easy to see that the natural inclusion of the above complex into $\mathcal{O} \xrightarrow{d} \Omega(n)$ induces an isomorphism in the first hypercohomology groups. Further, $\mathbb{H}^1(\mathcal{O} \xrightarrow{d} \Omega(n))$ is identified with the tangent space to $Conn^0_{X,p,n}$ at zero. The latter space is identified with the universal cover of $Conn^0_{X,p,n}$. □

Below we always assume the identification (7).

3.3. **Improving $\varphi_*\mathcal{D}^\bullet$.** According to Definition 1, in order to give rise to a pencil of connections, $\varphi_*\mathcal{D}^\bullet$ has to be isomorphic to $p_1^*\mathcal{W}$, where $\mathcal{W}$ is a vector bundle on $\mathbf{A}$. This is impossible, since the restriction of $\varphi_*\mathcal{D}^\bullet$ to $\{m\} \times \mathbb{P}^1$ is not a trivial bundle (see Lemma 1). This is easy to cure if $g = 0$: just twist by $\mathcal{O}_{\mathbb{P}^1}(-1)$. If $g > 0$ we need to choose a trivial subbundle in the restriction of $\varphi_*\mathcal{D}^\bullet$ to every $m \in \mathbf{A}$.

Let us now recall that we have chosen a space Δ of a-cycles on X. Note that every degree zero line bundle has a unique non-singular connection with trivial a-monodromy. This gives a splitting of the standard exact sequence

$$0 \to H^0(X, \Omega(n)) \to Conn^0_{X,p,n} \to Pic^0_X \to 0. \tag{8}$$

Thus, the choice of Δ gives a splitting

$$\varphi_*\mathcal{D}^\bullet|_{z=\infty} = H^0(X, \Omega(n)) \oplus H^1(X, \mathcal{O}). \tag{9}$$

We define $\mathcal{V}$ as the subsheaf of $\varphi_*\mathcal{D}^\bullet$ whose sections over $U \subset \mathbb{P}^1$ are given by

$$\Gamma(U, \mathcal{V}) = \{s \in \Gamma(U, \varphi_*\mathcal{D}^\bullet) : \ s|_{z=\infty} \in H^1(X, \mathcal{O})\}.$$

Proposition 1. *The bundle $\mathcal{V}$ is trivial.*

Proof. Note that the isomorphism (4) is not canonical but the inclusion of $\mathcal{V}' := \oplus_1^{g+n-1}\mathcal{O}_{\mathbb{P}^1}(1)$ into $\varphi_*\mathcal{D}^\bullet$ is canonical. It is easy to see that the proposition is equivalent to $\mathcal{V}'_\infty \cap H^1(X, \mathcal{O}) = 0$. Thus it is enough to prove that $\mathcal{V}'$ corresponds to the connections on the trivial bundle under (7). To prove the claim, we note that

$$\mathcal{V}'_\infty = \{s(\infty) | s \in H^0(\mathbb{P}^1, \varphi_*\mathcal{D}^\bullet), s(0) = 0\}.$$

Now the claim follows from the proof of Lemma 1 (see diagram (6) and two paragraphs after it). □

3.4. **Globalization.** So far we were working with a fixed quadruple (X, p, f_0, Δ). Let us now globalize the picture to $\hat{\mathbf{A}}$. First we get a vector bundle $\varphi_*\mathcal{D}^\bullet$ on $\mathbf{A} \times \mathbb{P}^1$ with a connection along $\mathbf{A}$ with a pole along $\mathbf{A} \times \{0\}$. The restriction of this connection to $\mathbf{A} \times \{z\}$ we denote by ∇_z. Modifying $\varphi_*\mathcal{D}^\bullet$ at $z = \infty$ as above we get a vector bundle $\mathcal{V}$ on $\hat{\mathbf{A}} \times \mathbb{P}^1$, its restriction to any point of $\hat{\mathbf{A}}$ being a trivial bundle on $\mathbb{P}^1$. Thus $\mathcal{V} = p_1^*\mathcal{W}$ for some vector bundle $\mathcal{W}$ on $\hat{\mathbf{A}}$.

Theorem 1. $(\mathcal{W}, \nabla)$ *is a pencil of connections on $\hat{\mathbf{A}}$ in the sense of Definition 1.*

Proof. As was explained in §3.1, $\varphi_*\mathcal{D}^\bullet$ is equipped with a connection ∇_z for $z \neq 0, \infty$, thus $\mathcal{V}$ also has a z-dependent connection. It is not hard to show directly that this connection extends to $z = \infty$ and has a simple pole at $z = 0$. However, in the next section we shall calculate both the residue of ∇_z at zero and $\lim_{z\to\infty} \nabla_z$ explicitly, completing the proof of the theorem. □

4. Identifying structures

We shall see shortly that Frobenius structures come from ∇_∞ and the residue of the pencil of connections. Then we shall calculate these parts of the pencil explicitly. It will yield Theorem 1. It will also follow that this pencil of connections gives rise to twisted Frobenius manifold structures on $\hat{\mathbf{A}}$.

4.1. **From pencils of connections to Frobenius manifolds and WDVV equation.** Let $(\mathcal{W}, \nabla)$ be a pencil of connections on a manifold $\mathbf{M}$. One interprets it as a family of flat connections on $\mathbf{M}$, parameterized by $\mathbb{P}^1 \setminus 0$. Our condition on the pole at zero implies that this family is of the form $\nabla_\infty + \Phi/z$, where z is the standard coordinate on $\mathbb{P}^1$, ∇_∞ is the restriction of ∇ to $\mathbf{M} \times \{\infty\}$, Φ is a Higgs field, that is, an $\mathcal{O}_{\mathbf{M}}$-*linear* map $\mathcal{W} \to \mathcal{W} \otimes \Omega_{\mathbf{M}}$.

Assume now that there exists a *primitive section*, i.e., a section ρ of $\mathcal{W}$ such that $\nabla_\infty \rho = 0$ and $\Phi(\rho, \cdot) : \mathcal{T}\mathbf{M} \to \mathcal{W}$ is an isomorphism. One uses this isomorphism to carry ∇_∞ and Φ to $\mathcal{T}\mathbf{M}$. The former becomes a flat structure $\tilde{\nabla}$ on $\mathbf{M}$ (it is automatically without torsion). The latter becomes a commutative associative multiplication in the tangent sheaf. We denote this multiplication by $\circ$. The equation $\Phi(\rho, e) = \rho$ defines a unit for $\circ$. One can show that $\circ$ does not depend on the choice of ρ (while $\tilde{\nabla}$ does depend).

The last ingredient needed to equip $\mathbf{M}$ with a structure of a Frobenius manifold is a symmetric bilinear product compatible with $\tilde{\nabla}$ and $\circ$. In other words, $\tilde{\nabla}$ is the Levi–Civita connection for this metric, while the multiplication operators are symmetric. These structures altogether make $\mathcal{T}\mathbf{M}$ into a sheaf of *Frobenius algebras*, so that $\mathbf{M}$ becomes a *Frobenius manifold*.

Set $a(z) = -z$, then giving such a metric is equivalent to a ∇-flat a-symmetric non-degenerate pairing $\langle \cdot, \cdot \rangle : \mathcal{V} \otimes a^*\mathcal{V} \to \mathcal{O}_{\mathbf{M} \times \mathbb{P}^1}$, where $\mathcal{V}$ is the pull-back of $\mathcal{W}$ to $\mathbf{M} \times \mathbb{P}^1$. Indeed, if s_1 and s_2 are sections of $\mathcal{W}$, then we have

$$\left\langle \left(\nabla_\infty + \frac{\Phi}{z} \right) s_1, s_2 \right\rangle + \left\langle s_1, \left(\nabla_\infty - \frac{\Phi}{z} \right) s_2 \right\rangle = d \left\langle s_1, s_2 \right\rangle .$$

This is equivalent to $\langle \nabla_\infty s_1, s_2 \rangle + \langle s_1, \nabla_\infty s_2 \rangle = d\langle s_1, s_2 \rangle$ and $\langle \Phi s_1, s_2 \rangle = \langle s_1, \Phi s_2 \rangle$.

One then chooses flat coordinates t_A on $\mathbf{M}$ and shows that there exists locally on $\mathbf{M}$ a *potential* F such that

$$\langle \partial_A \circ \partial_B, \partial_C \rangle = \tilde{\nabla}_{\partial_A} \tilde{\nabla}_{\partial_B} \tilde{\nabla}_{\partial_C} F.$$

The associativity condition for $\circ$ transforms into WDVV equation for F.

Remark. In the original definition of Frobenius manifold the structures above are required to be homogeneous, so the Euler field is added to the structures. Thus our definition is more general, in fact we shall construct non-homogeneous Frobenius manifolds.

4.1.1. *Constructing primitive sections.* Let m be a point of $\mathbf{M}$, $\rho_m \in \mathcal{W}_m$ be such that $\Phi(\rho_m, \cdot) : \mathcal{T}_m\mathbf{M} \to \mathcal{W}_m$ is an isomorphism. Then we can extend ρ_m to a ∇_∞-flat section ρ of $\mathcal{W}$. Unfortunately, ρ is defined on a universal cover $\tilde{\mathbf{M}}$ of $\mathbf{M}$, thus it gives rise to a Frobenius manifold structure on the open subset of $\tilde{\mathbf{M}}$, where $\Phi(\rho, \cdot)$ is an isomorphism. One easily checks that $\circ$ descends to $\mathbf{M}$. However, the metric and $\tilde{\nabla}$ do not descend to $\mathbf{M}$. In §6.2 we shall construct primitive sections on open subsets of $\mathbf{M}$ defined up to a scalar, then the flat structures also descend to these open subsets of $\mathbf{M}$.

4.2. **Isomonodromic deformations.** We are going to identify ∇_∞ with some isomonodromic deformation. Thus we shall need some generalities on isomonodromy. For more details we refer to [BBT]. Note that there is a standard relation between isomonodromy and Frobenius manifolds, see [Sab], but it will not be used in our paper.

Let $\mathbf{Y} \to \mathbf{M}$ be a holomorphic fiber bundle, equipped with a family of meromorphic flat connections on fibers (i.e., a relative connection). This family is said to be *isomonodromic* if it can be extended to a flat absolute meromorphic connection on $\mathbf{Y}$. It is known that the family of connections with regular singularities is isomonodromic if and only if the monodromy does not change.

4.2.1. *Universal isomonodromy for line bundles.* We shall need a baby version of isomonodromy, namely, isomonodromy for *line bundles* with singular connections. In this case a family of meromorphic connections is isomonodromic if and only if the monodromy does not change (even if connections have irregular singularities).

Let $\mathbf{M}_{g,n}$ be the moduli space of triples (X, p, x), where X is a curve (which we assume smooth complete over $\mathbb{C}$), $p \in X$, x is a coordinate to order n at p (that is, an n-jet of a coordinate). Let $Conn_n \to \mathbf{M}_{g,n}$ be the moduli space of line bundles with connections with a pole of order at most n at p and no other poles. One defines the *universal isomonodromic connection* on this fibration by the following requirements: a family is isomonodromic if (1) the monodromy representation is constant and (2) the x-expansion of the polar part of the connection at p does not change. The existence and uniqueness of such a connection follows easily from the Riemann–Hilbert correspondence.

4.3. **Identifying ∇_∞.** Let $m = (X, p, f_0) \in \mathbf{A}$. Let $\bar{\mathbf{A}}$ be the formal completion of $\mathbf{A}$ at m. Recall that in §4.2.1 we defined a fibration $Conn_n \to \mathbf{M}_{g,n}$. Let $Conn_n^0 \to \mathbf{M}_{g,n}$ be its part corresponding to degree zero line bundles and let $\widetilde{Conn}_n^0 \to \mathbf{M}_{g,n}$ be its relative universal cover. In other words,

$$\widetilde{Conn}_n^0 = \bigsqcup_{(X,p,x)} \widetilde{Conn}^0_{X,p,n},$$

where $\widetilde{Conn}^0_{X,p,n}$ is defined in §3.2. Consider the diagram

$$\begin{array}{ccc} \varphi_* \mathcal{D}^\bullet|_{\bar{\mathbf{A}} \times \{z=\infty\}} & \longrightarrow & \widetilde{Conn}_n^0 \\ \downarrow & & \downarrow \\ \bar{\mathbf{A}} & \longrightarrow & \mathbf{M}_{g,n} \end{array} \tag{10}$$

The lower map is given in the following way: f in the neighborhood of p gives a polar part of order n. This gives a coordinate to order n. In other words, the coordinate is $x = f^{-\frac{1}{n}}$. Precisely, this coordinate is defined up to a multiplication by a root of unity. We fix such a choice (this is why we restrict this map to $\bar{\mathbf{A}}$). It follows from Lemma 2 that this is a pull-back diagram. It follows from §4.2.1 that the right fibration in (10) is equipped with isomonodromic connection.

Lemma 3. *The upper map respects connections. Thus ∇_∞ is the pull-back of the isomonodromic connection.*

We delegate the proof to the Appendix.

We have calculated ∇_∞ on $\varphi_*\mathcal{D}^\bullet$. Now we want to understand ∇_∞ on $\mathcal{V}$. To avoid confusion, we shall denote the connection on $\mathcal{V}$ by $\nabla^{\mathcal{V}}$ until the end of this subsection. Clearly, $\nabla_z = \nabla^{\mathcal{V}}_z$ for $z \neq \infty$.

Lemma 4. (*a*) *$\nabla^{\mathcal{V}}_\infty$ respects splitting (9).*
(*b*) *The restriction of $\nabla^{\mathcal{V}}_\infty$ to the universal cover of the space of non-singular connections with trivial a-monodromy is the cover of the isomonodromic connection.*
(*c*) *The restriction of $\nabla^{\mathcal{V}}_\infty$ to the space of trivial bundles is described as follows: a family $(\mathcal{O}, d_X + \rho)$ is flat if and only if the a-periods of ρ are constant and the x-expansion of the polar part of ρ is constant.*

Proof. (b) Follows from the previous lemma because we do not modify $\varphi_*\mathcal{D}^\bullet$ along this subspace.
(a) By (b) it is enough to prove that the space of trivial bundles is preserved. It follows from the fact that the sections of $\mathcal{V}$ corresponding to trivial bundles at $z = \infty$ correspond to the sections of $\varphi_*\mathcal{D}^\bullet$ that vanish at $z = \infty$.
(c) Let $\mathcal{C} = (\mathcal{O}, d_X + \rho)$ be a family with constant a-monodromy and a constant x-expansion of the polar part. Let $\mathcal{C}'$ be a section of $\varphi_*\mathcal{D}^\bullet|_{z=\infty}$ covering a family of non-singular connections such that its a-monodromy is trivial and the b-monodromy is reciprocal to that of $\mathcal{C}$. Then the projection of $\mathcal{C}'' = \mathcal{C} + \mathcal{C}'$ to $Conn_n$ is isomonodromic in the sense of §4.2.1. Hence it can be extended to a ∇-flat section $\tilde{\mathcal{C}}''$ of $\varphi_*\mathcal{D}^\bullet$ in the neighborhood of $z = \infty$. Then $\tilde{\mathcal{C}}''/z$ is also ∇-flat. It can be viewed as a section of $\mathcal{V}$, equal to $(\mathcal{O}, d_X + \rho)$ at $z = \infty$. It remains to recall that $\nabla^{\mathcal{V}}_\infty$ is defined as $\lim_{z\to\infty} \nabla^{\mathcal{V}}_z$. □

4.4. **Deformation theory.** Denote the complex

$$0 \to \mathcal{T}(-1) \xrightarrow{\times\omega} \mathcal{O}(n) \to 0$$

by $\mathcal{K}^\bullet$ and take $m = (X, p, f_0) \in \mathbf{A}$. Then the restriction $\mathcal{K}^\bullet|_m$ governs deformations of m, where only deformations of f_0 that preserve periods are allowed. Thus $\mathcal{K}^\bullet|_m$ is the deformation complex of $\mathbf{A}$. We give more details in §7.3.

Remark. $\mathcal{K}^\bullet$ is actually a dg-Lie algebra. This gives a relation of our work to [BK], where a similar complex is used (in a higher-dimensional situation). See also [M].

Consider the residue of ∇ at $z = 0$, denote it by Φ. We want to give a cohomological interpretation of Φ. Consider a morphism of complexes

$$\begin{array}{ccccc} \mathcal{T}(-2) & \xrightarrow{\times\omega\oplus\times\omega} & \mathcal{O}(n-1)\oplus\mathcal{O}(n-1) & \xrightarrow{(\times\omega)-(\times\omega)} & \Omega(2n) \\ \downarrow & & \downarrow{\scriptstyle -} & & \downarrow{\scriptstyle =} \\ 0 & \longrightarrow & \mathcal{O}(n-1) & \xrightarrow{\times\omega} & \Omega(2n) \end{array} \tag{11}$$

The upper complex is $\mathcal{K}^\bullet \otimes (\mathcal{O}(-1) \xrightarrow{\times\omega} \Omega(n))$, while the lower complex is naturally quasi-isomorphic to $\mathcal{O}(-1) \xrightarrow{\times\omega} \Omega(n)$ shifted by one. Thus we get a map

$$\mathcal{T}\hat{\mathbf{A}} \otimes R^1\varphi_*(\mathcal{O}(-1) \xrightarrow{\times\omega} \Omega(n)) \to R^1\varphi_*(\mathcal{O}(-1) \xrightarrow{\times\omega} \Omega(n)).$$

Lemma 5. *This map coincides with Φ.*

Again we delegate the proof to the Appendix.

Notice that $\mathcal{K}^\bullet$ is exact except at zeros of $\omega = d_X f_0$, therefore for a fixed $m \in \hat{\mathbf{A}}$ the complex is naturally quasi-isomorphic to $\oplus_s \mathcal{O}_{q_s}$, where q_s are zeros of ω (if ω

has multiple zeros, they should be viewed as schemes with nilpotents). Similarly, $\mathcal{D}^\bullet|_{z=0}$ is quasi-isomorphic to $\oplus_s(\Omega_X)_{q_s}$.

It follows easily from the lemma above that under this identification Φ becomes the componentwise multiplication. Thus $\Phi(\rho_m,\cdot)$ is an isomorphism for a generic $\rho_m \in \mathcal{V}_m$. It follows that taking different primitive sections we can construct twisted Frobenius structures on open subsets covering $\hat{\mathbf{A}}$ (see §4.1).

The tangent sheaf multiplication also has a cohomological interpretation, namely, there is a natural map $\mathcal{K}^\bullet \otimes \mathcal{K}^\bullet \to \mathcal{K}^\bullet \otimes \mathcal{O}(n)$, similar to (11).

Lemma 6. *The induced map on cohomology coincides with $\circ$. In particular $\circ$ does not depend on the choice of a primitive section.*

Proof. Follows from associativity of cohomological multiplication. □

One would like to have a canonical choice of a primitive section; this will be discussed in §6.

4.5. **Metric.** We shall sketch a construction of a bilinear product on $\mathcal{V}$. We start with the natural $\varphi^{\cdot}\mathcal{O}_{\hat{\mathbf{A}}\times\mathbb{P}^1}$-linear map of complexes on $\mathbf{X}\times\mathbb{P}^1$

$$\mathcal{D}^\bullet(1-n)\otimes a^*\mathcal{D}^\bullet \to (\mathcal{O}(-n-1)\xrightarrow{d}\Omega)\boxtimes\mathcal{O}_{\mathbb{P}^1}(1).$$

For $z\neq\infty$ it gives rise to a ∇-flat map: $\langle\cdot,\cdot\rangle : \varphi_*\mathcal{D}^\bullet\otimes a^*\varphi_*\mathcal{D}^\bullet \to \mathcal{O}_{\mathbb{P}^1}(1)$ (it is a priori singular at $z=\infty$ because $\mathcal{D}^\bullet(1-n)$ is not quasi-isomorphic to $\mathcal{D}^\bullet$ at $z=\infty$). This form is *a-skew-symmetric*.

We claim that the restriction of this metric to $\mathcal{V}$ vanishes at $z=\infty$. Indeed, let s_1, s_2 be sections of $\mathcal{V}$. Then, by definition of $\mathcal{V}$, $s_i|_{z=\infty}$ are tangent vectors to the space of non-singular connection with zero a-periods. One can check that our form agrees with the natural symplectic form on the space of connections. Notice that the symplectic form on non-singular connections can be identified with the intersection form on $H^1(X,\mathbb{C})$. It remains to recall that Δ is isotropic.

Thus composing $\langle\cdot,\cdot\rangle|_{\mathcal{V}}$ with multiplication by z we get an *a-symmetric* bilinear form on $\mathcal{V}$. We leave it to the reader to check that the form is non-degenerate.

Remark. We have to multiply by z because we identified $\mathcal{O}_{\mathbb{P}^1}(1)$ with functions with a pole at $z=0$ in (2).

4.6. **Relation with the construction of Krichever.** The above Frobenius structure is equivalent to that of [K1, K2]. To give a bridge between these papers and ours we present here a different point of view on $\mathcal{V}$.

Fix $m=(X,p,f_0,\Delta)\in\hat{\mathbf{A}}$ and fix some set of disjoint closed curves a_i ($i=1,\dots,g$) on X representing a-cycles. Assume that $p\notin\cup a_i$. Let χ be a 1-form on X with the only pole of order at most n at p and *jumps* $\lambda_i\omega$ along a_i, where $\lambda_i\in\mathbb{C}$ ($\omega=d_Xf$ as before).

Lemma 7. *For every $z\in\mathbb{P}^1$ there is a natural isomorphism between the set of such forms with jumps and $\mathcal{V}|_{m,z}$.*

Proof. Let us first consider $z\neq\infty$. The idea is that a form with jumps can be, in some sense, viewed as a Čech cocycle of $\mathcal{D}^\bullet$.

Precisely, we take a cover $\mathcal{U}_j$ of X such that there is a unique j with $p\in\mathcal{U}_j$. Let (α_j,s_{jk}) be a cocycle (cf. §7.1), representing a class in $\mathbb{H}^1(X,\mathcal{D}^\bullet)$ so that

$\alpha_j \in \Gamma(\mathcal{U}_j, \Omega(n))$, $s_{jk} \in \Gamma(\mathcal{U}_j \cap \mathcal{U}_k, \mathcal{O})$. The cocycle conditions are

$$\alpha_j - \alpha_k = d_X s_{jk} + \frac{\omega}{z} s_{jk}, \qquad s_{jk} + s_{kl} + s_{lj} = 0, \qquad s_{jk} = -s_{kj}.$$

Hence s_{jk} represents a class in $H^1(X, \mathcal{O})$. It is easy to check that for every such a cocycle s_{jk} we can find holomorphic functions h_j *with constant jumps along* a_i*'s* such that $s_{jk} = h_j - h_k$. This functions are unique if we require $h_j(p) = 0$, if $p \in \mathcal{U}_j$. It follows that

$$\alpha_j - d_X h_j - \frac{\omega}{z} h_j$$

patch together to a 1-form α with jumps along a_i. We leave it to the reader to check that this map from the cohomology group to forms with jumps is an isomorphism.

For $z = \infty$ one gives an isomorphism by the following conditions

(1) If $\lambda_i = 0$ for all i, then χ corresponds to $(\mathcal{O}, d + \chi)$.

(2) If χ has no pole at p, then consider the local system given by the transition functions $\exp(\lambda_i)$ (in particular its a-monodromy is trivial). This map to the space of local systems on X is then lifted to a map to $\mathcal{V}|_{z=\infty}$. □

The reader should compare this description with Lemma 4. It follows that in terms of forms with jumps ∇_∞ can be described very easily: a family is flat if a-periods do not change, λ_i's do not change and the x-expansion of χ at p does not change.

Krichever fixes a basis in $H_1(X, \mathbb{Z})$ and a choice of $f^{-1/n}$. It allows him to construct a flat basis in the space of forms with jumps (denoted by Ω_a). See §7.5 and §7.2 of [K1].

5. Multi-point generalizations

In this section we generalize the above constructions to the case of abelian integrals with many poles. To define such an integral we need an extra "reference point".

Let X be as before, $p_0, \dots, p_k$ be distinct points of X. Let $n_1, \dots, n_k$ be non-negative integers and consider the divisors $\mathfrak{D} = p_0 + \sum_{i=1}^k n_i p_i$, $\mathfrak{D}' = \sum_{i=1}^k p_i$; we assume that $\sum n_i + k \geq 2$, $k \geq 1$. By an *abelian integral* with polar divisor $\mathfrak{D}$ we mean a function f on the universal cover of (X, p_0) such that df descends to a meromorphic differential on X with a pole of order exactly $n_i + 1$ at p_i for $1 \leq i \leq k$ and no other poles (note that there is no pole at p_0). Let $\mathbf{A}_{g,n_1,\dots,n_k}$ be the corresponding moduli space of collections $(X, p_0, \dots, p_k, f)$.

For fixed $(X, p_1, \dots, p_k)$ we get a period map $H_1(X \setminus \{p_1, \dots, p_k\}, \mathbb{Z}) \to \mathbb{C}$. As before, identifying the homology groups for nearby curves, we get a period foliation on $\mathbf{A}_{g,n_1,\dots,n_k}$. Fix one of the leaves $\mathbf{A}$.

Remark. Note that $\mathbf{A}_{g,n_1,\dots,n_k} = \mathbf{A}' \times \mathbb{C}$, where $\mathbf{A}'$ is the corresponding moduli space of abelian differentials. The splitting is given by the abelian integrals, which vanish at p_0.

For $(X, p_0, \dots, p_k, f) \in \mathbf{A}$ we have a natural map $H_1(X \setminus \{p_1, \dots, p_k\}, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ and we pull the intersection form back via this map to a degenerate alternating form on $H_1(X \setminus \{p_1, \dots, p_k\}, \mathbb{Z})$. Let $\Delta \subset H_1(X \setminus \{p_1, \dots, p_k\}, \mathbb{Z})$ be maximal isotropic with respect to this form. Then Δ has rank $g+k-1$. Denote by $\hat{\mathbf{A}}$ the module space of collections $(X, p_0, \dots, p_k, f, \Delta)$, where $(X, p_0, \dots, p_k, f) \in \mathbf{A}$, Δ is as above.

Again we denote by $\varphi : \mathbf{X} \to \mathbf{A}$ the natural projection from the universal curve, by $\tilde{p}_i$ the universal divisors on $\mathbf{X}$ corresponding to p_i. Set $\tilde{\mathfrak{D}} = \tilde{p}_0 + \sum_1^k n_i \tilde{p}_i$, $\tilde{\mathfrak{D}}' = \sum_1^k \tilde{p}_i$. Analogously to (2), we set $\omega = d_X f$ and consider the complex

$$\mathcal{D}^\bullet := \left(\mathcal{O}(-\tilde{\mathfrak{D}}') \boxtimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{d_X + \frac{\omega}{z}} \Omega(\tilde{\mathfrak{D}}) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \right).$$

Similarly to Lemma 1, we prove that the restriction of $\varphi_* \mathcal{D}^\bullet$ to a fixed point of $\mathbf{A}$ is isomorphic to the vector bundle $\left(\oplus_1^{g+k-1} \mathcal{O}_{\mathbb{P}^1} \right) \oplus \left(\oplus_1^{g+\sum n_i} \mathcal{O}_{\mathbb{P}^1}(1) \right)$.

Similarly to §4.2.1 we introduce the moduli space $\mathbf{M}_{g;n_1,\ldots,n_k}$ parameterizing collections $(X, p_0, \ldots, p_k, x_1, \ldots, x_k)$, where $X, p_0, \ldots, p_k$ are as before, x_i is a coordinate to order n_i at p_i. Note that if $n_i = 0$, then x_i is just the point p_i itself.

Let $Conn^0_{n_1,\ldots,n_k} \to \mathbf{M}_{g;n_1,\ldots,n_k}$ be the moduli space of degree zero line bundles trivialized over $\mathfrak{D}'$ with connections with a pole bounded by $\mathfrak{D}$. Let $\widetilde{Conn}^0_{n_1,\ldots,n_k} \to \mathbf{M}_{g;n_1,\ldots,n_k}$ be the relative universal cover.

Let $m \in \mathbf{A}$. Let $\bar{\mathbf{A}}$ be the formal completion of $\mathbf{A}$ at m. We have

Lemma 8. *Let ω be a meromorphic form on a disc with a pole of order $n \geq 2$ at the center. Then there is a unique up to multiplication by a root of unity $n-1$-jet of coordinate x such that $\omega = (x^{-n} + ax^{-1})\, dx + O(1)$, where $O(1)$ is a regular form.*

Proof. Let ω_0 be a form with a simple pole and the same residue as ω. Then

$$x = c \left(\int (\omega - \omega_0) \right)^{\frac{-1}{n-1}},$$

where c is a suitable constant. □

Applying this Lemma at every p_i with $n_i > 0$, we get a map $\psi : \bar{\mathbf{A}} \to \mathbf{M}_{g;n_1,\ldots,n_k}$.

Similarly to (10) we have a pull-back diagram

$$\begin{array}{ccc} \varphi_* \mathcal{D}^\bullet|_{\bar{\mathbf{A}} \times \{z=\infty\}} & \longrightarrow & \widetilde{Conn}^0_{n_1,\ldots,n_k} \\ \downarrow & & \downarrow \\ \bar{\mathbf{A}} & \xrightarrow{\psi} & \mathbf{M}_{g;n_1,\ldots,n_k} \end{array}$$

Again the right fibration is equipped with isomonodromic connection and the upper map respects connections. Thus ∇_∞ is the pull-back of the isomonodromic connection. As before we get a splitting

$$\varphi_* \mathcal{D}^\bullet|_{\bar{\mathbf{A}} \times \{z=\infty\}} = \widetilde{Bun} \oplus Conn_{triv},$$

where $\widetilde{Bun}$ is the relative universal cover of the moduli space of bundles with trivialization on $\mathfrak{D}'$ and non-singular connections with zero a-periods, and $Conn_{triv}$ is the moduli space of connections on the trivial bundle with standard trivialization on $\mathfrak{D}'$. Then we define $\mathcal{V}$ as the vector bundle on $\hat{\mathbf{A}}$ whose sheaf of sections is

$$\{ s \in \varphi_* \mathcal{D}^\bullet : s|_{z=\infty} \in \widetilde{Bun} \}.$$

We have a multi-pole version of Theorem 1: $\mathcal{V} = p_1^* \mathcal{W}$ for some vector bundle $\mathcal{W}$ on $\hat{\mathbf{A}}$ and $(\mathcal{W}, \nabla)$ is a pencil of connections on $\hat{\mathbf{A}}$ in the sense of Definition 1. Finally, the multiplication and the metric have a cohomological description similar to the single-point case.

6. Primitive sections

6.1. **A general strategy.** Let $g \geq 0$ and $2 \leq k \leq n$ be integers. We start with a complete smooth curve Y of genus g, a point $q \in Y$ and differentials $\omega_0 \in H^0(Y, \Omega_Y((n+1)q))$ and $\rho_0 \in H^0(Y, \Omega_Y(kq))$ having no common zeros. Then choose an abelian integral f_0 of ω_0, and consider the local coordinate $x = f_0^{-1/n}$ near q.

Now consider the moduli space $\mathbf{A}'$, parameterizing collections (X, p, f, Δ, ρ), where $X \ni p$ is a complete curve of genus g, f is an abelian integral such that $df \in H^0(X, \Omega_X((n+1)p))$, f has the same periods as f_0 (more precisely, the collection is in the same leaf of the period foliation, cf. §2.2), Δ is maximal isotropic in $H_1(X, \mathbb{Z})$, $\rho \in H^0(X, \Omega_X(kp))$ such that ρ and ρ_0 have the same a-periods, and the coefficients of the $f^{-1/n}$-expansion of ρ at p are equal to the coefficients of the x-expansion of ρ_0 at q. By Lemma 4(c), ρ is ∇_∞-flat. Let $\mathbf{A}''$ be the non-empty open subset of $\mathbf{A}'$, where ρ and ω have no common zeros. Then ρ is a primitive section on $\mathbf{A}''$, and $\mathbf{A}''$ is a Frobenius manifold. We leave the multipoint generalization to the reader.

6.2. **Another choice of a primitive section.** The following choice of a primitive section is often used in the theory of integrable systems. Let $\hat{\mathbf{A}}$ be as in §2.2. We use notation of §2.3. Let k be an integer such that $2 \leq k \leq n$. Consider a section ρ_k of $\Omega(k)$ with the following properties

(1) Its polar part at $\tilde{p}$ is of the form $x^{-k}dx$, where $x = f^{-1/n}$.

(2) The a-periods of ρ_k are zero.

This form is defined locally over $\hat{\mathbf{A}}$ up to a multiplication by an nth root of unity. It follows from Lemma 4(c) that $(\mathcal{O}, d_X + \rho)$ is a flat section of $\mathcal{V}|_{z=\infty}$. This is a primitive section on the open set $\hat{\mathbf{A}}_k \subset \hat{\mathbf{A}}$, where ρ_k and ω have no common zeros. Thus it gives rise to a connection on $\hat{\mathbf{A}}_k$, which depends only on k. The metric is defined up to multiplication by a root of unity.

It is curious that if n is even, then $\rho_{n/2}$ is defined up to sign, so the corresponding metric is also defined canonically.

Unfortunately, we do not know whether $\hat{\mathbf{A}}_k$ is always non-empty. We have the following partial result

Lemma 9. (*a*) *For a generic leaf* $\mathbf{A} \subset \mathbf{A}_{g,n}$ *all the sets* $\hat{\mathbf{A}}_k$ *are non-empty.*
(*b*) *If* $n > k + 2g$, *then* $\hat{\mathbf{A}}_k$ *is non-empty.*

Proof. (a) It is enough to show that for a generic $(X, p, f) \in \mathbf{A}_{g,n}$ the corresponding ρ and df have no common zeros. Consider first the case $g > 0$. Changing f to $f + \int \alpha$, where α is any holomorphic differential on X does not change the k-jet of the coordinate $x = f^{-1/n}$. Thus it does not change ρ, so it is enough to show that there is $\alpha \in H^0(X, \Omega_X)$ such that $df + \alpha$ and ρ have no common zeros. Let $\{q_i\}$ be the set of zeros of ρ, let H_i be the set of holomorphic differentials α such that $(df + \alpha)(q_i) = 0$. Since H_i is an affine subspace of $H^0(X, \Omega_X)$, it is enough to show that $H_i \neq H^0(X, \Omega_X)$ (because finite number of proper affine subspaces cannot cover a vector space). Thus we have to show that the natural map $H^0(X, \Omega_X(-q_i)) \to H^0(X, \Omega_X)$ is not an isomorphism. Using a standard exact sequence we see, that it is equivalent to $H^1(X, \Omega_X(-q_i)) = \mathbb{C}$, which, by Serre duality, is equivalent to $H^0(X, \mathcal{O}_X(q_i)) = \mathbb{C}$. The last equality follows from $g > 0$.

Now assume that $g=0$, so that $X=\mathbb{P}^1$, and we can assume $p=\infty$. Since $\mathbf{A}_{0,n}$ is irreducible, it is enough to find a single f such that df and ρ have no common zeros. We can take $f=z^n+z$, where z is the standard coordinate. Then an easy calculation shows that $\rho=-z^{k-2}\,dz$.

(b) Let $\mathbf{A}$ be any leaf, $(X,p,f)\in\mathbf{A}$. If we add a section of $H_0(X,\mathcal{O}_X((n-k)p))$ to f, the k-jet of $f^{-1/n}$ does not change, so ρ does not change, hence, as before, it is enough to show that there is $g\in H_0(X,\mathcal{O}_X((n-k)p))$ such that $df+dg$ and ρ have no common zeros. The rest of the proof is as above. In the last step we use that a sheaf of degree at least $2g-1$ has zero first cohomology group. $\square$

7. Appendix: Some proofs

7.1. **Calculating $\varphi_*\mathcal{D}^\bullet$.** We start with the Čech calculation of $\varphi_*\mathcal{D}^\bullet$. Pick $m=(X,p,f)\in\mathbf{A}$, let $\bar{\mathbf{A}}$ be the completion of $\mathbf{A}$ at m. Consider the following cover of X: $X=\dot{X}\cup D$, where $\dot{X}=X\setminus p$, D is the formal neighborhood of p. Let $\dot{D}=\dot{X}\cap D$ be the punctured formal neighborhood. Let $\bar{\mathbf{X}}$ be the restriction of $\mathbf{X}$ to $\bar{\mathbf{A}}$. Since affine schemes have no infinitesimal deformations, $\bar{\mathbf{X}}$ can be covered by $\dot{X}\times\bar{\mathbf{A}}$ and $D\times\bar{\mathbf{A}}$. To calculate $\varphi_*\mathcal{D}^\bullet$ we shall use the Čech resolution of $\mathcal{D}^\bullet$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}(-1) & \xrightarrow{d_X+\frac{\omega}{z}} & \Omega(n) & \longrightarrow & 0 & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & C^0(\mathcal{O}(-1)) & \longrightarrow & \begin{smallmatrix}C^1(\mathcal{O}(-1))\oplus\\ C^0(\Omega(n))\end{smallmatrix} & \longrightarrow & C^1(\Omega(n)) & \longrightarrow & 0
\end{array}$$

Consider a section of $R^1\varphi_*\mathcal{D}^\bullet$ over $\bar{\mathbf{A}}$. It is represented by a family of 1-cocycles (α_1,α_2,s). Precisely, α_1 is a relative 1-form on $\dot{X}\times\bar{\mathbf{A}}\to\bar{\mathbf{A}}$, α_2 is a relative 1-form on $D\times\bar{\mathbf{A}}\to\bar{\mathbf{A}}$ with a pole of order at most n along $\{p\}\times\bar{\mathbf{A}}$, $s\in\mathcal{O}_{\dot{D}\times\bar{\mathbf{A}}}$. The cocycle condition is

$$\alpha_1-\alpha_2=d_Xs+\frac{\omega s}{z}. \tag{12}$$

We extend $d_X+\frac{\omega}{z}$ to an absolute connection $\nabla=d+\frac{df}{z}$ as in §2.4.

We want to describe the Gauss–Manin connection on the first hypercohomology sheaf directly. First, we define a map

$$\Psi_1:\wedge^2\Omega_{\dot{X}\times\bar{\mathbf{A}}}\to\varphi^*\Omega_{\bar{\mathbf{A}}}\otimes\Omega_{\dot{X}\times\bar{\mathbf{A}}/\bar{\mathbf{A}}}.$$

It is defined as follows: there is a natural surjective map $\varphi^*\Omega_{\bar{\mathbf{A}}}\otimes\Omega_{\dot{X}\times\bar{\mathbf{A}}}\to\wedge^2\Omega_{\dot{X}\times\bar{\mathbf{A}}}$ (recall that $\dim X=1$). Thus we can lift a section of $\wedge^2\Omega_{\dot{X}\times\bar{\mathbf{A}}}$ to $\varphi^*\Omega_{\bar{\mathbf{A}}}\otimes\Omega_{\dot{X}\times\bar{\mathbf{A}}}$ and then project it to $\varphi^*\Omega_{\bar{\mathbf{A}}}\otimes\Omega_{\dot{X}\times\bar{\mathbf{A}}/\bar{\mathbf{A}}}$. The lift is defined up to an element of the second symmetric power of $\varphi^*\Omega_{\bar{\mathbf{A}}}$, thus the projection does not depend on the lift. Similarly, we can define maps

$$\Psi_2:\wedge^2\Omega_{D\times\bar{\mathbf{A}}}\to\varphi^*\Omega_{\bar{\mathbf{A}}}\otimes\Omega_{D\times\bar{\mathbf{A}}/\bar{\mathbf{A}}}.$$

$$\Psi_{12}:\wedge^2\Omega_{\dot{D}\times\bar{\mathbf{A}}}\to\varphi^*\Omega_{\bar{\mathbf{A}}}\otimes\Omega_{\dot{D}\times\bar{\mathbf{A}}/\bar{\mathbf{A}}}.$$

Now let us lift α_i to an absolute 1-form $\tilde{\alpha}_i$ such that $\tilde{\alpha}_2$ has a pole of order at most n on $\{p\}\times\bar{\mathbf{A}}$. It follows from (12) that

$$\tilde{\alpha}_1-\tilde{\alpha}_2=\nabla s+\beta, \tag{13}$$

where β is a section of $\varphi^*\Omega_{\bar{\mathbf{A}}}$. The Gauss–Manin connection is given by

$$\nabla(\alpha_1,\alpha_2,s)=(\Psi_1(\nabla\tilde{\alpha}_1),\Psi_2(\nabla\tilde{\alpha}_2),\beta). \tag{14}$$

Lemma 10. *(1) The 1-cochain (14) is a cocycle for* $\mathcal{D}^\bullet \otimes \mathcal{O}(n+1) \otimes \varphi^*\Omega_{\bar{\mathbf{A}}}$.
(2) If we change the lifts $\alpha_i \rightsquigarrow \tilde{\alpha}_i$*, then (14) changes by a coboundary.*
(3) ∇ *satisfies the Leibnitz rule with respect to multiplication of cocycles by functions on* $\bar{\mathbf{A}}$.

Proof. We have a well-defined relative differential $d_X : \varphi^*\Omega_{\bar{\mathbf{A}}} \to \varphi^*\Omega_{\bar{\mathbf{A}}} \otimes \Omega_{X\times\bar{\mathbf{A}}/\bar{\mathbf{A}}}$. The proof of (1) and (2) is based on the following: *if* γ *is a section of* $\varphi^*\Omega_{\bar{\mathbf{A}}}$*, then* $\Psi_i(\nabla\gamma) = d_X\gamma + \frac{1}{z}\gamma \otimes \omega$. The proof of this statement is left to the reader, let us prove (1). It follows from (13) that $\nabla\tilde{\alpha}_1 - \nabla\tilde{\alpha}_2 = \nabla\beta$. Now let us apply Ψ_{12} to both sides, it gives $\nabla\alpha_1 - \nabla\alpha_2 = d_X\beta + \frac{1}{z}\beta \otimes \omega$ and the cocycle condition follows.
We leave the proofs of (2) and (3) to the reader. □

It follows that ∇ descends to a connection on $\varphi_*\mathcal{D}^\bullet$. A little difficulty is that we get a cocycle of $\mathcal{D}^\bullet \otimes \mathcal{O}(n+1) \otimes \varphi^*\Omega_{\bar{\mathbf{A}}}$ instead of that of $\mathcal{D}^\bullet \otimes \varphi^*\Omega_{\bar{\mathbf{A}}}$. Fortunately, there is a natural quasi-isomorphism (compare with §3.1)

$$\mathcal{D}^\bullet \hookrightarrow \mathcal{D}^\bullet \otimes \mathcal{O}(n+1) \tag{15}$$

for $z \neq \infty$. For $z = \infty$ one needs to extend the cocycle to a neighborhood of $z = \infty$ first, apply ∇, and then pass to the limit as $z \to \infty$ to evaluate ∇_∞. This will be done in the next subsection.

7.2. **Proof of Lemma 3.** Consider an isomonodromic family of connections on $\bar{\mathbf{A}}$ given by a family of cocycles $(\alpha_1, \alpha_2, \exp(s))$ as in §7.1 (now we have $z = \infty$). Here $\exp(s)$ is a cocycle defining the line bundle. It has a single-valued logarithm because the bundle has degree zero. We need to show that this family is ∇_∞-flat. The cocycle condition (12) becomes

$$\alpha_1 - \alpha_2 = d_X s.$$

Since the family is isomonodromic, the forms α_i can be extended to absolute *closed* forms $\tilde{\alpha}_i$. The condition that the x-expansion of the polar part does not change in the family can be written in the following way

$$\tilde{\alpha}_2 = h(x^{-1})dx + \gamma(\varepsilon, x), \tag{16}$$

where ε is a coordinate on $\bar{\mathbf{A}}$, h is a polynomial with constant coefficients, γ is a 1-form regular on $\{p\} \times \bar{\mathbf{A}}$. A simple calculation in local coordinates shows that, changing the lift $\alpha_2 \rightsquigarrow \tilde{\alpha}_2$ if necessary, we can assume that γ/x has a logarithmic pole on $\{p\} \times \bar{\mathbf{A}}$.

Since $f = x^{-n}$, we have $h(x^{-1})dx \wedge df = 0$. Now we want to extend the cocycle to the neighborhood of $z = \infty$. To this end we write $\omega s = \sigma_1 - \sigma_2$, where σ_1 is a relative 1-form on $\dot{X} \times \bar{\mathbf{A}}$, σ_2 is a relative 1-form on $D \times \bar{\mathbf{A}}$ with at most simple pole on $\{p\} \times \bar{\mathbf{A}}$. This is always possible, since $H^1(X, \Omega_X(p)) = 0$. It is easy to see that $(\alpha_1 + \frac{\sigma_1}{z}, \alpha_2 + \frac{\sigma_2}{z}, s)$ satisfies the cocycle condition (12).

We extend σ_i to an absolute form $\tilde{\sigma}_i$. One checks that we can choose $\tilde{\sigma}_2$ so that it has at most simple pole on $\{p\} \times \bar{\mathbf{A}}$ and $\tilde{\sigma}_2 \wedge df = 0$. Using (16) we get

$$\nabla\left(\tilde{\alpha}_2 + \frac{\tilde{\sigma}_2}{z}\right) = \left(d + \frac{df}{z}\right)\left(\tilde{\alpha}_2 + \frac{\tilde{\sigma}_2}{z}\right) = \frac{df}{z} \wedge \gamma + \frac{d\tilde{\sigma}_2}{z} = O\left(\frac{1}{z}\right).$$

A similar argument, shows that $\nabla\left(\tilde{\alpha}_1 + \frac{\tilde{\sigma}_1}{z}\right) = O(1/z)$. Applying Ψ we see that

$$\nabla\left(\alpha_1 + \frac{\sigma_1}{z}, \alpha_2 + \frac{\sigma_2}{z}, s\right) = (0, 0, \beta) + O\left(\frac{1}{z}\right), \tag{17}$$

for some β. Note also that since γ/x has logarithmic pole on $\{p\} \times \bar{\mathbf{A}}$, (17) is a cocycle of $\mathcal{D}^\bullet$ (a priori it is a cocycle of $\mathcal{D}^\bullet(n)$). Therefore we do not need to invert the quasi-isomorphism (15) (which could have altered the behavior at $z = \infty$). Taking limit as $z \to \infty$, we get a cocycle of the form $(0, 0, \beta)$. The cocycle condition shows that $d_X\beta = 0$, and we see that this cocycle is a coboundary, so the family (α_1, α_2, s) is a flat section of ∇_∞.

7.3. **Proof of Lemma 5.** Let us fix $m = (X, p, f_0, \Delta) \in \hat{\mathbf{A}}$ and give an explicit description of the map $\mathbb{H}^1(\mathcal{K}^\bullet|_m) \to \mathcal{T}_m\hat{\mathbf{A}}$. Consider a cocycle (h_1, h_2, τ) of $\mathcal{K}^\bullet|_m$, where $h_1 \in \mathcal{O}_{\dot{X}}$, $h_2 \in \mathcal{O}_D(n)$, $\tau \in \mathcal{T}(\dot{D})$. Then τ represents a class of $H^1(X, \mathcal{T})$, thus it gives rise to an infinitesimal family of curves $\hat{X} \to \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$. A function $\hat{f}$ on $\hat{X}$ is given by the conditions: $\hat{f}|_{\dot{X}\times \operatorname{Spec}\mathbb{C}[\varepsilon]/\varepsilon^2} = f_0 + \varepsilon h_1$, $\hat{f}|_{D\times \operatorname{Spec}\mathbb{C}[\varepsilon]/\varepsilon^2} = f_0 + \varepsilon h_2$. The periods of $\hat{f}$ are constant in the family because h_1 is a single-valued function on $\dot{X}$. Thus we have assigned a vector of $\mathcal{T}_m\hat{\mathbf{A}}$ to (h_1, h_2, τ). We leave it to the reader to check that it indeed gives an isomorphism of $R^1\varphi_*\mathcal{K}^\bullet$ with the tangent sheaf of $\hat{\mathbf{A}}$.

Let us take a Čech cocycle (α_1, α_2, s) of $\mathcal{D}^\bullet|_{m,z=0}$. We have $\alpha_1 \in \Omega_{\dot{X}}$, $\alpha_2 \in \Omega_D(n)$, $s \in \mathcal{O}_{\dot{D}}$. Let us also take a cocycle (h_1, h_2, τ) of $\mathcal{K}^\bullet|_m$, where $h_1 \in \mathcal{O}_{\dot{X}}$, $h_2 \in \mathcal{O}_D(n)$, $\tau \in \mathcal{T}(\dot{D})$. This cocycle represents some $\xi \in \mathcal{T}_m\hat{\mathbf{A}}$. The product of cocycles is given by $(h_1\alpha_1, h_2\alpha_2, \tau\alpha_1 + h_2 s)$. The cocycle condition for (α_1, α_2, s) is given by $\alpha_1 - \alpha_2 = \omega s$, hence s is uniquely determined by α_1 and α_2. Again, we view $\bar{\mathbf{X}}$ as glued from $\dot{X} \times \bar{\mathbf{A}}$ and $D \times \bar{\mathbf{A}}$. We extend α_1 to a 1-form $\tilde{\alpha}_1$ on $\dot{X} \times \bar{\mathbf{A}}$ using this direct product structure. Similarly we extend α_2 to $\tilde{\alpha}_2$ and h_i to $\tilde{h}_i$. Let us lift ξ to a vector field $\tilde{\xi}_1$ along $\dot{X} \times \{m\}$ and to $\tilde{\xi}_2$ along $D \times \{m\}$. In particular we have $\langle \tilde{\xi}_i, \tilde{\alpha}_i \rangle = 0$.

Unwinding the definition of ∇, we see that Φ is given on cocycles by the formula

$$\Phi(\xi, (\alpha_1, \alpha_2, s)) = (\langle \xi, \Psi_1(df \wedge \tilde{\alpha}_1)\rangle, \langle \xi, \Psi_2(df \wedge \tilde{\alpha}_2)\rangle, ?).$$

Since the last entry of the cocycle is uniquely determined by the others, it is enough to prove that

$$\langle \xi, \Psi_i(df \wedge \tilde{\alpha}_i)\rangle = h_i\alpha_i.$$

Using the identification of cohomology of $\mathcal{K}^\bullet|_m$ with the tangent space of $\hat{\mathbf{A}}$ at m, one finds that $\langle \tilde{\xi}_i, df\rangle = \partial_{\tilde{\xi}_i}(f) = h_i$. It follows that

$$\langle \xi, \Psi_i(df \wedge \tilde{\alpha}_i)\rangle = \langle \tilde{\xi}_i, df \wedge \tilde{\alpha}_i\rangle = \langle \tilde{\xi}_i, df\rangle\tilde{\alpha}_i - \langle \tilde{\xi}_i, \tilde{\alpha}_i\rangle df = h_i\tilde{\alpha}_i.$$

The first equality is a "consistency property" of Ψ_i.

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